EEE 203, HW 1   NAME:_____SOLUTIONS___

Problem 1.
Consider the signal *x(t)* whose graph is shown below. Sketch the following signals:
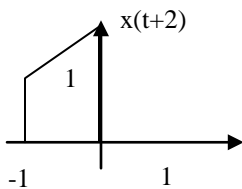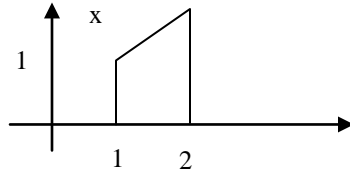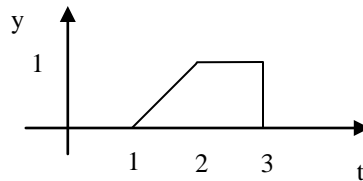*x(t+2), x(2-t), RT₁[x],* where R denotes the reflection operation and T_{t0} denotes shift
delay operation by t0.
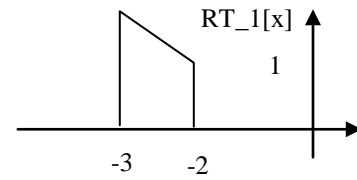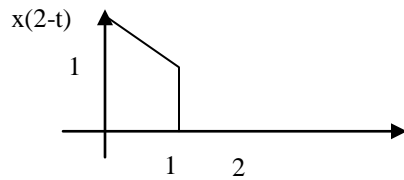
Problem 2.
Describe the following signals in terms of elementary functions (δ, *u* ,*r*, …) and compute
$\int_{-\infty}^{\infty} x(t)\delta(t-3)dt$ and $\int_{-\infty}^{\infty} y(t)\delta(t-2)dt$.
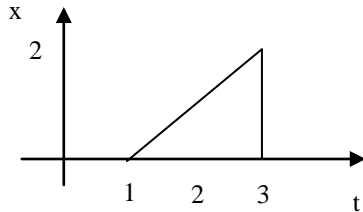
$$x(t) = r(t-1) - r(t-3) - 2u(t-3)$$
$$\int x(t)\delta(t-3)dt = \frac{[x(3^+) + x(3^-)]}{2} = 1$$
$$y(t) = r(t-1) - r(t-2) - u(t-3)$$
$$\int y(t)\delta(t-2)dt = \frac{[y(2^+) + y(2^-)]}{2} = 1$$

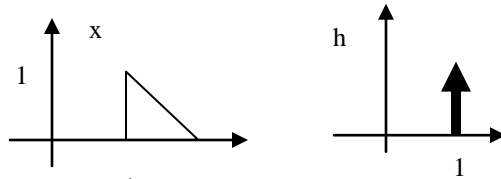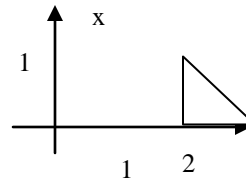**Problem 1.** Compute the convolution $h*x$ when $x(t) = u(t\text{-}1)\text{-}r(t\text{-}1)+r(t\text{-}2)$, $h(t)=\delta(t\text{-}1)$.

$(u(t)$ is the unit step, $r(t)$ is the unit ramp)

$(h * x)(t) = u(t - 2) - r(t - 2) + r(t - 3)$, or

Problem 2. Consider the filters

1. $y[t] = x(t + 1) - x(t - 1)$
2. $y(t) = \int_{-\infty}^{t-1} e^{-\tau+t}x(\tau)d\tau$

1. Find and graph their impulse responses.
      1. $h(t) = \delta(t + 1) - \delta(t - 1)$
      2. $y(t) = \int_{-\infty}^{t-1} e^{-\tau+t}\delta(\tau)d\tau = \int_{-\infty}^{t-1} e^{-0+t}\delta(\tau)d\tau = e^t \int_{-\infty}^{t-1} \delta(\tau)d\tau = e^t u(t - 1)$

2. Find and graph their step responses.
      1. $h(t) = u(t + 1) - u(t - 1)$
      2. $y(t) = \int_{-\infty}^{t-1} e^{-\tau+t}u(\tau)d\tau = e^t u(t - 1) \int_0^{t-1} e^{-\tau}d\tau = [e^t - e]u(t - 1)$

3. Which filters are causal? (Justify)
      1. Non-causal because $h(t) \neq 0$, for some t < 0.
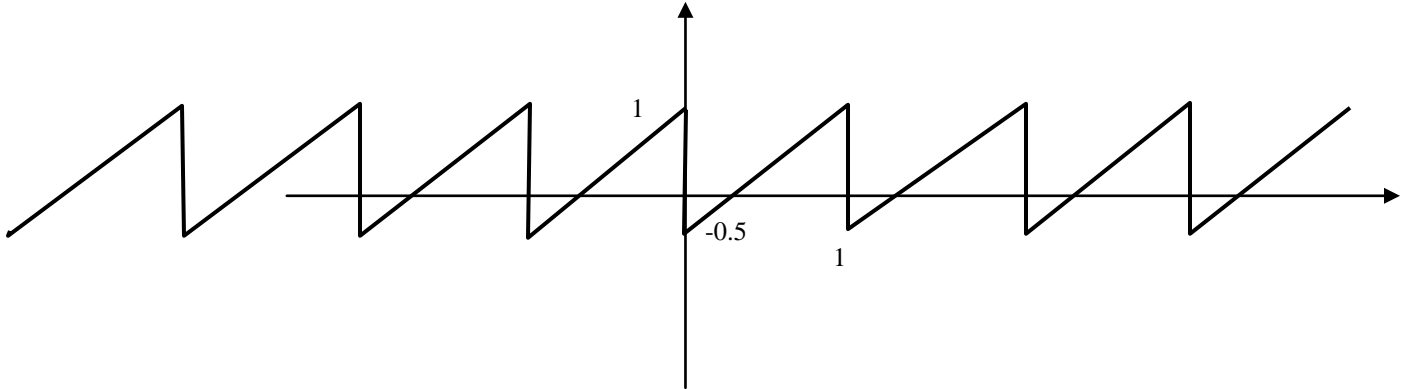   2. Causal because h(t) = 0, for all t < 0.

4. Which filters are stable? (Justify)
      1. Stable because $\int |h| = 2 < \infty$
      2. Unstable because $\int |h|$ diverges.
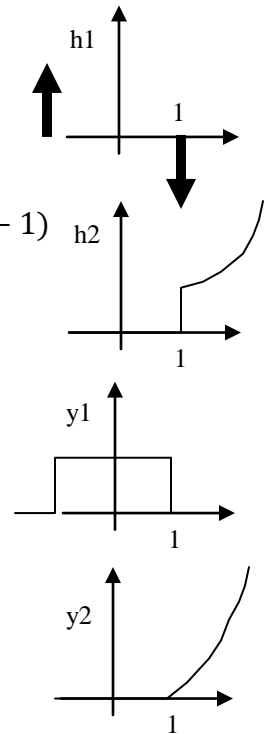
Problem 1:   Let $x(t)$ be the periodic signal shown in the figure below (sawtooth wave with offset).
Compute the coefficients a_k of the Fourier series expansion of $x(t)$.

The derivative of x is $\frac{dx}{dt} = 1.5 - \sum_n 1.5\delta(t-n)$. From the tables, the Fourier series (FS) coefficients of the impulse train are $a_k = -1.5(1), \forall k$. Then, the FS coefficients of x, say b_k are given by $b_k = \frac{1}{jkw_0}a_k, \ k \neq 0, \ w_0 = \frac{2\pi}{1}$. Thus, $b_k = \frac{-1.5}{jk2\pi}, \ k \neq 0$.

For k = 0, we compute the FS coefficient directly from the definition, $b_0 = \frac{1}{T}\int_T x(t)dt = -\frac{1}{12} + \frac{2}{6} = \frac{3}{12} \Rightarrow b_0 = \frac{1}{4}$

Problem 2: Let $X(jw)$ be the Fourier transform of $x(t) = e^{-|t-2|}$. Find $X(j0)$ and $\int_{-\infty}^{\infty} wX(jw)dw$.

From the definition of the Fourier transform and its inverse, $X(j0) = \int x(t)e^{-j0t}dt = \int e^{-|t-2|}dt \Rightarrow$

$X(j0) = 2$.

On the other hand, $\frac{dx}{dt}(0) = \frac{1}{2\pi}\int (jwX(jw))e^{jw0}dw \Rightarrow \int wX(jw)dw = -2\pi j\frac{dx}{dt}(0) \Rightarrow$

$\int wX(jw)dw = -2\pi je^{-2}$

Problem 3:

Consider the filter with impulse response $h(t) = e^{-(t-1)}u(t-1)$.

1. Find the transfer function and sketch the Bode Plot

2. Find the Fourier transform of the output when $x(t) = e^{-t}u(t)$ and when $x(t) = \sin(t)$

3. Find the output when $x(t) = e^{-t}u(t)$ (a) using convolution, and (b) taking the inverse Fourier transform of your answer to part 2.

1. $H(jw) = e^{-jw}F\{e^{-t}u(t)\} = \dfrac{e^{-jw}}{jw+1}$

$|H(jw)| = \dfrac{1}{\sqrt{w^2 + 1^2}}$, $\angle H(jw) = -\tan^{-1}\left(\dfrac{w}{1}\right) - w$ (corner frequency at 1)

In MATLAB,
>> H=tf(1,[1 1])
>> H.iodelay=1
>> bode(H)

Bode Diagram

2.
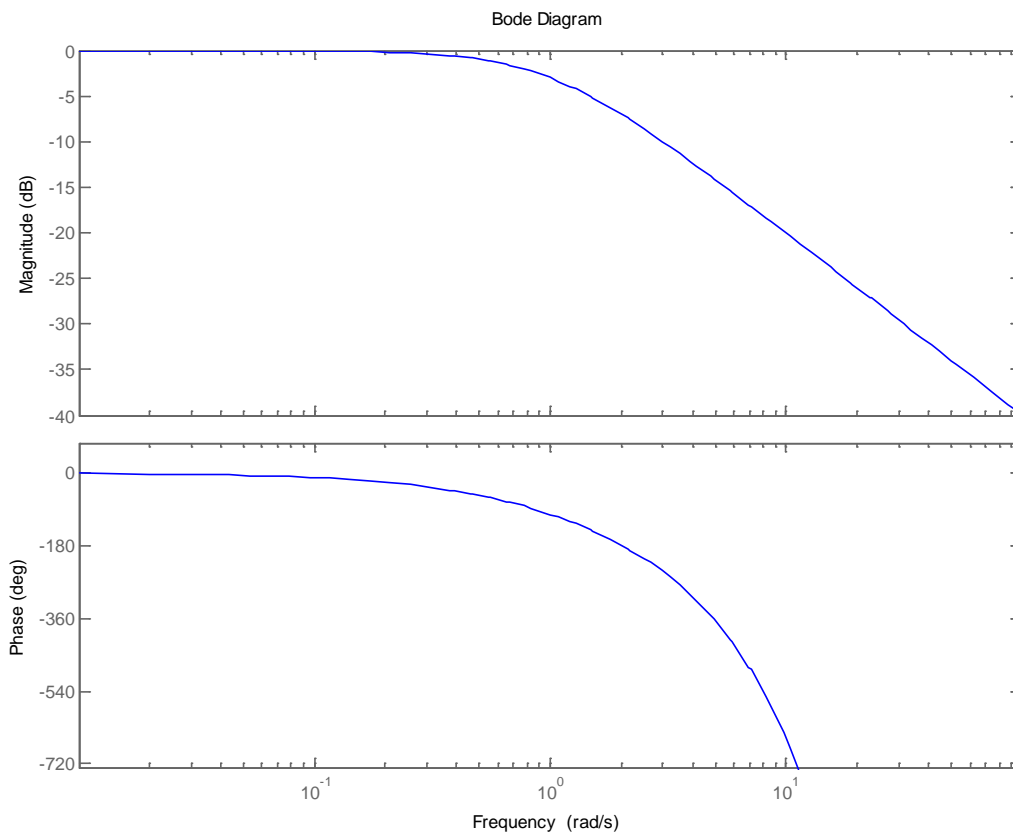
Exponential input : $F\{x\} = \dfrac{1}{jw+1}$, $Y(jw) = H(jw)X(jw) = \dfrac{e^{-jw}}{(jw+1)^2}$

Sinusoid input : $F\{x\} = F\{\sin t\} = \dfrac{\pi}{j}[\delta(w-1) - \delta(w+1)]$, $Y(jw) = H(jw)X(jw) =$

$$= \frac{\pi}{j}\left[\frac{e^{-j}}{j+1}\delta(w-1) - \frac{e^{j}}{-j+1}\delta(w+1)\right] \text{ (evaluate H(jw) at the sin frequency)}$$

3. $\displaystyle y(t) = (h*x)(t) = \int_{-\infty}^{\infty} e^{-(t-1-\tau)}u(t-1-\tau)e^{-\tau}u(\tau)d\tau = e^{-(t-1)}u(t-1)\int_{0}^{t-1} e^{0}d\tau = (t-1)e^{-(t-1)}u(t-1)$

$$= F^{-1}\{Y(jw)\} = F^{-1}\left\{\frac{e^{-jw}}{(jw+1)^2}\right\} = F^{-1}\left\{\frac{1}{(jw+1)^2}\right\}\Big|_{t-1} = te^{-t}u(t)\Big|_{t-1} = (t-1)e^{-(t-1)}u(t-1)$$

Problem 1:
Find the largest sampling interval T_s to allow perfect reconstruction of the signals:
(NOTE: h*x denotes convolution of h and x)

1. $\dfrac{\sin 2t}{t} * \dfrac{\sin 3t}{t}$

2. $\dfrac{\sin t}{t} + \dfrac{\sin 2t}{2t}$

3. $\dfrac{\sin t}{t} * \sin 7t$

Solutions using either the shortcut estimates of the max frequency or a full computation of the Fourier transform of the composite signals are acceptable.

1. Using the shortcuts, $w_m = min(w_{m1}, w_{m2}) = 2\dfrac{rad}{s} \Rightarrow T_s < \dfrac{\pi}{2}s$. The same answer is obtained by performing the Fourier transform of the composite signal (multiplication of two pulses in the frequency domain).

2. Using the shortcuts, $w_m = max(w_{m1}, w_{m2}) = 2\dfrac{rad}{s} \Rightarrow T_s < \dfrac{\pi}{2}s$. The same answer is obtained by performing the Fourier transform of the composite signal (summation of two pulses in the frequency domain).

3. Using the shortcuts, $w_m = min(w_{m1}, w_{m2}) = 1\dfrac{rad}{s} \Rightarrow T_s < \dfrac{\pi}{1}s$. This is only a conservative estimate in this case (i.e., this sample time is guaranteed to work but larger times are possible). By performing the Fourier transform of the composite signal (multiplication of a pulse and two impulses in the frequency domain) we have no overlap between the two signals and a product which is identically zero. Hence, the actual upper bound for T_s is infinity (i.e., any sampling time would work).

Problem 2:
For a sampling process with sampling time 1ms, what is the cutoff frequency of the ideal low-pass filter needed to avoid any aliasing effects and what is the ideal low pass filter for reconstruction?

Sampling at 1ms, i.e., a frequency of 1000Hz, allows recovery of signals with maximum frequency $500Hz = 500(2\pi) = 3142\dfrac{rad}{s}$. This should be the cutoff frequency of the ideal low-pass filter used to pre-process the analog signal to avoid aliasing (anti-aliasing filter). It should also be the cutoff frequency of the ideal low-pass filter used for reconstruction of the maximum possible range of signals. The reconstruction filter can be defined by:

- Its frequency response $H(jw) = \begin{cases} T(= 1e-3), \ for \ |w| < 3142\dfrac{rad}{s} \ (or \ 500Hz) \\ 0 \ otherwise \end{cases}$ or,

- Its impulse response $h(t) = F^{-1}\{H(jw)\} = \dfrac{0.001 \sin(3142t)}{\pi t}$

(Either one is acceptable).

EEE 203, HW 6   NAME:_____

Problem 1: Consider the continuous time causal filter with transfer function

$$H(s) = \frac{s+1}{(s+2)(s-1)}$$

Compute the response of the filter to $x[t] = u[t]$

$$y(t) = L^{-1}\left\{\frac{s+1}{(s+2)(s-1)s}\right\} = L^{-1}\left\{\frac{\frac{-1}{6}}{(s+2)} + \frac{\frac{4}{6}}{(s-1)} + \frac{\frac{-3}{6}}{s}\right\}$$

$$= \frac{-1}{6}e^{-2t}u(t) + \frac{4}{6}e^{t}u(t) + \frac{-3}{6}u(t)$$

Problem 2: Consider the continuous time causal filter described by the differential equation

$$\frac{d^2y}{dt^2} + 1\frac{dy}{dt} + 1y = x$$

Compute the steady-state response of the filter to $x[t] = cos[1t + 1]u(t)$.

$$H(s) = \frac{1}{s^2 + s + 1} = \frac{1}{(s + 0.5 + j0.866)(s + 0.5 - j0.866)}$$

The system is stable (t.f. poles in the LHP), hence the steady-state response is well-defined.

$$y_{ss}(t) = |H(j1)|\cos(1t + 1 + \angle[H(j1)]) =$$

$$= \frac{1}{|j^2 + j + 1|}\cos\left(t + 1 - \text{atan}\frac{1}{0}\right) =$$

$$= 1\cos(t - 32.7^o)$$

Problem 1: Consider the causal filter described by the difference equation

$$y[n] = \frac{1}{3}y[n-1] + \frac{1}{2}x[n-1]$$

1. Determine the transfer function
2. Compute the response of the filter to $x[n] = u[n]$
3. Compute the steady state response to $x[n] = u[n]$
4. Compute the steady state response to $x[n] = \sin(\,^{n\pi}/_{16})\,u[n-1]$

Note: For a stable system if x converges to a periodic signal x_s, then y converges to a periodic signal that is the system response to the x_s. This can be computed using Fourier theory. A useful simplification is:

For Continuous Time: $\quad x(t) = e^{j\omega_o t} \Rightarrow y(t) = H(j\omega_o)e^{j\omega_o t}$

$$x(t) = \cos(\omega_o t) \Rightarrow y(t) = |H(j\omega_o)|\cos(\omega_o t + \angle H(j\omega_o))$$

For Discrete Time: $\quad x(n) = e^{j\Omega_o n} \Rightarrow y(n) = H(e^{j\Omega_o})e^{j\Omega_o n}$

$$x(n) = \cos(\Omega_o n) \Rightarrow y(n) = |H(e^{j\Omega_o})|\cos(\Omega_o n + \angle H(e^{j\Omega_o}))$$

where, $H(s), H(z)$ are the continuous and discrete system transfer functions, respectively.

1. $H(z) = \dfrac{\frac{1}{2}z^{-1}}{1-\frac{1}{3}z^{-1}} = \dfrac{\frac{1}{2}}{z-\frac{1}{3}}$

2. $y(z) = H(z)\dfrac{z}{z-1} = \dfrac{\frac{1}{2}z}{(z-\frac{1}{3})(z-1)} = \dfrac{-\frac{3}{4}z}{(z-\frac{1}{3})} + \dfrac{\frac{3}{4}z}{(z-1)} \Rightarrow y(n) = Z^{-1}\left\{\dfrac{-\frac{3}{4}z}{(z-\frac{1}{3})} + \dfrac{\frac{3}{4}z}{(z-1)}\right\} =$

$$y(n) = -\frac{3}{4}\left(\frac{1}{3}\right)^n u(n) + \frac{3}{4}u(n) \Rightarrow \begin{bmatrix} n \\ y_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0.5 & 0.667 & \cdots \end{bmatrix}$$

3. The filter is stable since the pole (1/3) has magnitude less than one. Hence, the steady-state response is well defined. For the constant steady-state, $y(n) = H(z=1)\cos 0n = \frac{3}{4}$

4. $y_{ss}(n) = |H(e^{j\Omega})|\sin\left(\Omega n + \angle H(e^{j\Omega})\right),\quad \Omega = \frac{\pi}{16}$

Then, $y_{ss}(n) = \dfrac{\frac{1}{2}}{\sqrt{\left\{\left[\cos\left(\frac{\pi}{16}\right)-\frac{1}{3}\right]^2 + \left[\sin\left(\frac{\pi}{16}\right)\right]^2\right\}}} \sin\left(\frac{\pi}{16}n - \text{atan}\,\dfrac{\sin\left(\frac{\pi}{16}\right)}{\cos\left(\frac{\pi}{16}\right)-\frac{1}{3}}\right) = 0.74\sin\left(\frac{\pi}{16}n - 17^o\right)$

Problem 2: Consider the discrete time causal filter with transfer function

$$H(z) = \frac{z-1}{(z-0.8)(z+0.9)}$$

Compute the response of the filter to $x[n] = u[n]$

$$y(z) = H(z)\frac{z}{z-1} = \frac{z}{(z-0.8)(z+0.9)}$$

$$y(n) = Z^{-1}\left\{z\left(\frac{\frac{0.8}{1.7}}{(z-0.8)} + \frac{\frac{0.9}{1.7}}{(z+0.9)}\right)\right\} = Z^{-1}\left\{\frac{0.47z}{(z-0.8)} + \frac{0.53z}{(z+0.9)}\right\}$$

$$y(n) = (0.47)0.8^n u(n) + (0.53)(-0.9)^n u(n) \Rightarrow \begin{bmatrix} n \\ y_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -0.1 & 0.73 & -0.15 & 0.54 \end{bmatrix}$$

EEE 203, Test 1　　　　　　　　　　　NAME:___Solutions____

Problem 1.
Consider the signal *x(t)* whose graph is shown below. Sketch the following signals:
x(1-t), $T_1R[x]$, where R denotes the reflection operation and T_{t0} denotes shift delay operation by t0.

1 ▲ x

　　1　2

1 ▲ x(1-t)

　1　2

1 ▲ R[x]

　1　2

1 ▲ $T_1R[x]$

　1　2
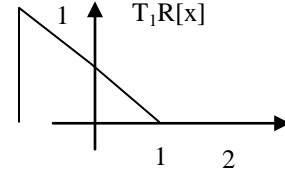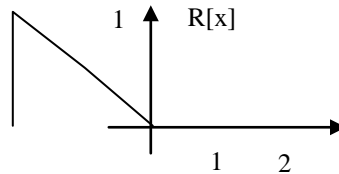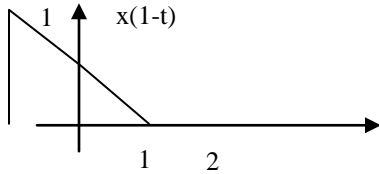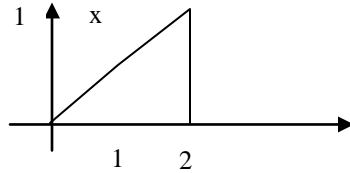
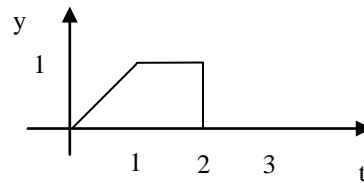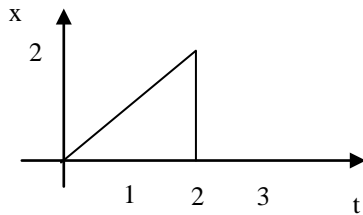Type equation here.

Problem 2.
Describe the following signals in terms of elementary functions (δ, *u* ,*r*, …) and compute
$\int_{-\infty}^{\infty} x(t)\delta(t-3)dt$ and $\int_{-\infty}^{\infty} y(t)\delta(t-3)dt$.

x
2

　1　2　3　　t

y
1

　1　2　3　　t

$$x(t) = r(t) - r(t-2) - 2u(t-2),\qquad \int x(t)\delta(t-3)dt = x(3) = 0$$
$$y(t) = r(t) - r(t-1) - u(t-2),\qquad \int y(t)\delta(t-3)dt = y(3) = 0$$

Problem 1. Compute the convolution $h*x$ when $h(t) = u(t-1)$, $x(t) = u(t+1) - u(t+2)$.

We have shown that $u * u = r$. Then, by time invariance,
$$h * x = u(t-1) * u(t+1) = u(t-1) * u(t+2) = r(t) - r(t+1)$$

Problem 2.
Consider the filters:

A. $y(t) = \int_{t-1}^{t} x(\tau)d\tau$

B. $y(t) = \int_{-\infty}^{t} x(\tau+1)d\tau$

1. Find and graph their impulse responses.

A. $h(t) = \int_{t-1}^{t} \delta(\tau)d\tau = \int_{-\infty}^{t} \delta(\tau)d\tau - \int_{-\infty}^{t-1} \delta(\tau)d\tau = u(t) - u(t-1)$
B. $h(t) = \int_{-\infty}^{t} \delta(\tau+1)d\tau = \int_{-\infty}^{t+1} \delta(\tau)d\tau = u(t+1)$

2. Which filters are causal? (Justify)

A is causal because h(t) = 0 for t<0.
B is not causal because h(t) assumes nonzero values for some t<0.

3. Which filters are stable? (Justify)

A is stable because $\int |h| = 1 < \infty$
B is not causal because h(t) is not absolutely integrable $\int |h| = r(t+1)$.

Problem 1: Consider the filter with impulse response $h(t) = e^{-t}u(t-1)$.

1. Find the Fourier transform of the output Y(jw) when $x(t) = e^{-2t}u(t)$

2. Find the time-domain expression for the output y(t) when $x(t) = e^{-2t}u(t)$

$$H(jw) = F\{e^{(-t-1+1)}u(t-1)\} = \frac{1}{e}F\{e^{-(t-1)}u(t-1)\} = \frac{e^{-jw}}{e}F\{e^{-t}u(t)\} = \frac{e^{-jw}}{e(jw+1)};$$

$$X(jw) = \frac{1}{jw+2} \Rightarrow Y(jw) = H(jw)X(jw) = \frac{e^{-jw}}{e(jw+1)(jw+2)}$$

$$Y(jw) = \frac{e^{-jw}}{e(jw+1)(jw+2)} \xrightarrow{PFE} \frac{e^{-jw}}{e}\left(\frac{1}{(jw+1)} + \frac{-1}{(jw+2)}\right)$$

$$\Rightarrow y(t) = \frac{1}{e}F^{-1}\left\{\left(\frac{1}{(jw+1)} + \frac{-1}{(jw+2)}\right)\right\}_{t-1} = \frac{1}{e}\{(e^{-t}u(t) - e^{-2t}u(t))\}_{t-1}$$

$$= \frac{1}{e}\left(e^{-(t-1)}u(t-1) - e^{-2(t-1)}u(t-1)\right)$$

Problem 2: Consider the continuous time causal filter described by the differential equation $\dfrac{dy}{dt} + 3y = ax$,

where a is an adjustable parameter.
1. Find the magnitude and phase of the frequency response of the filter.
2. Compute the value of a so that steady-state response of the filter to $x[t] = cos[t]u(t)$ has amplitude 1.

$$H(s) = \frac{a}{s+3}, \qquad |H(jw)| = \frac{|a|}{\sqrt{w^2+3^2}}, \qquad \angle H(jw) = \angle a - \tan^{-1}\frac{w}{3}$$

The steady state response is $y(t) = |H(j1)|\cos(t + \angle H(j1))$. For the amplitude to be 1, we should have
$$a = \sqrt{1+9} = \sqrt{10}$$

Problem 3: Let $x(t)$ be the periodic signal shown in the figure below (square wave with offset).
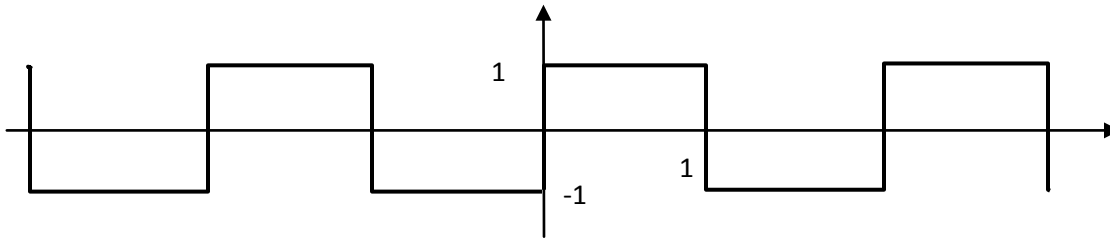
1. Compute the coefficients a_k of the Fourier series expansion of $x(t)$.
2. Compute the coefficients b_k of the output of an ideal lowpass filter H with cutoff frequency 1.5π, and input x.
The standard square wave in the tables has $x_o(t) \underset{FS}{\leftrightarrow} a_k = \frac{\sin kw_0 T_1}{k\pi}$

1. In terms of the standard square wave, $w_0 = \pi$, $T_1 = 0.5$

$$x(t) = 2x_o(t-0.5) - 1 \underset{FS}{\leftrightarrow} a_k = 2e^{-\frac{jk\pi}{2}}\frac{\sin\frac{k\pi}{2}}{k\pi}; \quad a_0 = 0$$

2. By the filtering property, $b_k = H(jkw_0)a_k = H(jk\pi)a_k$. Since the ideal low pass is 1 for $|w| < 1.5\pi, or\ |k| <$ 1.5, and 0 otherwise, we have

$$b_k = 2e^{-\frac{jk\pi}{2}}\frac{\sin\frac{k\pi}{2}}{k\pi}, for\ k = \pm1; \ b_k = 0\ otherwise$$

Problem 1:
Find the largest sampling interval T_s to allow perfect reconstruction of the signals:
(NOTE: h*x denotes convolution of h and x)

1. $\frac{\sin(t)}{t} * e^{-t}u(t)$.

2. $\frac{\sin(t)}{t}\cos t$

Using the shortcuts:
For convolution, $f_{Nyq} = \min(f_{Nyq1}, f_{Nyq2})$. The first signal is bandlimited to 1 rad/s, ($w_{Nyq1} = 2$)and the second is not bandlimited ($w_{Nyq2} = \infty$). Hence, $w_{Nyq} = 2, T_s \leq \frac{2\pi}{2} = \pi$.

For modulation, $f_{Nyq} = (f_{Nyq1} + f_{Nyq2})$. The first signal is bandlimited to 1 rad/s, ($w_{Nyq1} = 2$)and the second is bandlimited to 1 rad/s ($w_{Nyq2} = 2$). Hence, $w_{Nyq} = 4, T_s \leq \frac{2\pi}{4} = \frac{\pi}{2}$.

Problem 2:
The frequency spectrum of a vibration signal is shown in the figure below. Determine the sampling time constraints to avoid aliasing.
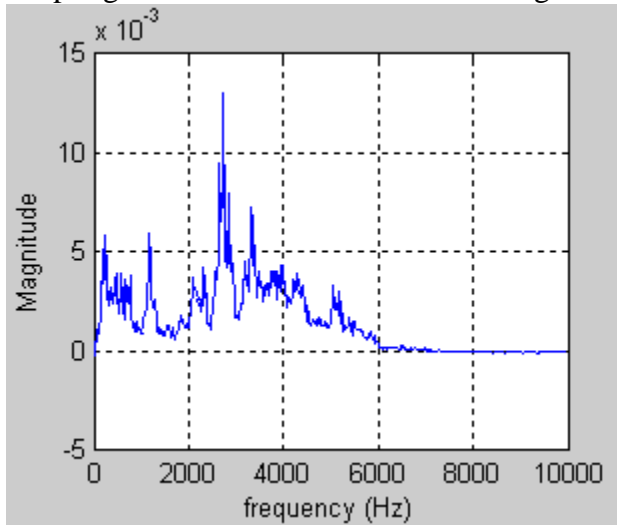
The signal is practically bandlimited to 7kHz (approximately). Hence, $T_s \leq \frac{1}{14k} = 0.071ms$.

CLOSED BOOK & NOTES. TRANSFORM TABLES ALLOWED. 30'

Problem 1: Consider the continuous time causal filter with transfer function

$$H(s) = \frac{s}{(s+2)(s+1)}$$

Compute the response of the filter to x(t) = u(t).

$$y(t) = L^{-1}\left\{\frac{s}{(s+2)(s+1)s}\right\} = L^{-1}\left\{\frac{-1}{(s+2)} + \frac{1}{(s+1)} + \frac{0}{s}\right\}$$

$$= -1e^{-2t}u(t) + 1e^{-t}u(t)$$

Problem 2: Consider the continuous time causal filter described by the differential equation

$$2\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2x + 2\frac{dx}{dt}$$

Compute the steady-state response of the filter to x(t) = sin(2t)u(t) + u(t).

$$H(s) = \frac{2s+2}{2s^2 + 2s + 1} = \frac{s+1}{(s+0.5+j0.5)(s+0.5-j0.5)}$$

The system is stable (t.f. poles in the LHP), hence the steady-state response is well-defined.

$$y_{ss}(t) = |H(j2)|\sin(2t + \angle[H(j2)]) + H(j0)(1) =$$

$$= \frac{2|j2+1|}{|2j^2 2^2 + 2j2 + 1|}\sin\left(2t + \text{atan}\frac{2}{1} - \text{atan}\frac{4}{-7} - 180^o\right) + \frac{2}{1} =$$

$$= \frac{2\sqrt{4+1}}{\sqrt{49+16}}\sin\left(2t + \text{atan }2 + \text{atan}\frac{4}{7} - 180^o\right) + 2 = \sqrt{\frac{4}{13}}\sin\left(2t - 86.8^o\right) + 2$$

$$= 0.55\sin\left(2t - 86.8^o\right) + 2$$

Problem 1: Consider the causal filter described by the difference equation

$$y[n] = \frac{1}{2} y[n-1] + \frac{1}{4} x[n-1]$$

1. Determine the transfer function
2. Compute the response of the filter to $x[n] = u[n]$
3. Compute the response to $x[n] = \delta[n]$
4. Compute the steady state response to $x[n] = \sin(^{n\pi}/_{16})\, u[n\text{-}1]$

Note: For a stable system if x converges to a periodic signal x_s, then y converges to a periodic signal that is the system response to the x_s. This can be computed using Fourier theory. A useful simplification is:

For Continuous Time:
$$x(t) = e^{j\omega_o t} \Rightarrow y(t) = H(j\omega_o) e^{j\omega_o t}$$
$$x(t) = \cos(\omega_o t) \Rightarrow y(t) = | H(j\omega_o) | \cos(\omega_o t + \angle H(j\omega_o))$$

For Discrete Time:
$$x(n) = e^{j\Omega_o n} \Rightarrow y(n) = H(e^{j\Omega_o}) e^{j\Omega_o n}$$
$$x(n) = \cos(\Omega_o n) \Rightarrow y(n) = | H(e^{j\Omega_o}) | \cos(\Omega_o n + \angle H(e^{j\Omega_o}))$$

where, $H(s), H(z)$ are the continuous and discrete system transfer functions, respectively.

1. $H(z) = \dfrac{\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-1}} = \dfrac{\frac{1}{4}}{z-\frac{1}{2}}$

2. $y(z) = H(z)\dfrac{z}{z-1} = \dfrac{\frac{1}{4}z}{\left(z-\frac{1}{2}\right)(z-1)} = z\left(\dfrac{-\frac{1}{2}}{\left(z-\frac{1}{2}\right)} + \dfrac{\frac{1}{2}}{(z-1)}\right) \Rightarrow y(n) = Z^{-1}\left\{\dfrac{-\frac{1}{2}z}{\left(z-\frac{1}{2}\right)} + \dfrac{\frac{1}{2}z}{(z-1)}\right\} =$

$$y(n) = -\frac{1}{2}\left(\frac{1}{2}\right)^n u(n) + \frac{1}{2}u(n) \Rightarrow \begin{bmatrix} n \\ y_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0.25 & 0.375 \end{bmatrix} \cdots$$

3. $y(z) = H(z)(1) = \dfrac{\frac{1}{4}}{\left(z-\frac{1}{2}\right)} \Rightarrow y(n) = Z^{-1}\left\{\dfrac{\frac{1}{4}z}{\left(z-\frac{1}{2}\right)}\right\}_{n \leftarrow n-1} =$

$$y(n) = \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u(n-1) \Rightarrow \begin{bmatrix} n \\ y_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0.25 & 0.125 \end{bmatrix} \cdots$$

4. The filter is stable since the pole (1/2) has magnitude less than one. Hence, the steady-state response is well defined and $y_{ss}(n) = |H(e^{j\Omega})| \sin\left(\Omega n + \angle H(e^{j\Omega})\right)$, $\Omega = \frac{\pi}{16}$

Then, $y_{ss}(n) = \dfrac{\frac{1}{4}}{\sqrt{\left\{\left[\cos\left(\frac{\pi}{16}\right)-\frac{1}{2}\right]^2 + \left[\sin\left(\frac{\pi}{16}\right)\right]^2\right\}}} \sin\left(\frac{\pi}{16}n - \operatorname{atan}\dfrac{\sin\left(\frac{\pi}{16}\right)}{\cos\left(\frac{\pi}{16}\right)-\frac{1}{2}}\right) = 0.482 \sin\left(\frac{\pi}{16}n - 22.1^o\right)$

Problem 2: Consider the discrete time causal filters with transfer functions

$$H_1(z) = \frac{z-1}{(z-0.9)(z+0.8)}, \quad H_2(z) = \frac{z-1}{(z+0.9)(z+1.8)}$$

1. Determine their stability properties.
2. Compute the response of the filters to $x[n] = u[n]$

1. The poles of H1 are 0.9, -0.8, both are less than 1 in magnitude, so the system is stable. The poles of H2 are -0.9, -1.8, the second has magnitude greater than 1 so the system is unstable.

2. $y_1(z) = H_1(z)\dfrac{z}{z-1} = \dfrac{z}{(z-0.9)(z+0.8)} = z\left(\dfrac{\frac{1}{1.7}}{(z-0.9)} + \dfrac{\frac{1}{-1.7}}{(z+0.8)}\right) \Rightarrow y_1(n) = Z^{-1}\left\{\dfrac{\frac{1}{1.7}z}{(z-0.9)} + \dfrac{\frac{1}{-1.7}z}{(z+0.8)}\right\} =$

$\frac{1}{1.7}(0.9)^n u(n) - \frac{1}{1.7}(-0.8)^n u(n) = [\,0, \ 1, \ 0.1, \ 0.73, \ 0.145, \ 0.5401, ... \,]$

2. $y_2(z) = H_2(z)\dfrac{z}{z-1} = \dfrac{z}{(z+0.9)(z+1.8)} = z\left(\dfrac{\frac{1}{0.9}}{(z+0.9)} + \dfrac{\frac{1}{-0.9}}{(z+1.8)}\right) \Rightarrow y_1(n) = Z^{-1}\left\{\dfrac{\frac{1}{0.9}z}{(z+0.9)} + \dfrac{\frac{1}{-0.9}z}{(z+1.8)}\right\} =$

$\frac{1}{0.9}(-0.9)^n u(n) - \frac{1}{0.9}(-1.8)^n u(n) = [0, \ 1, -2.7, \ 5.67, \ -10.935, ...]$

Problem 1: Consider the continuous time causal filter with transfer function
$$H(s) = \frac{s+2}{(s+1)(s-2)}$$
Compute the response of the filter to $x(t) = u(t)$

$$y(t) = L^{-1}\left\{\frac{s+2}{(s+1)(s-2)s}\right\} = L^{-1}\left\{\frac{4/6}{(s-2)} + \frac{1/3}{(s+1)} + \frac{-1}{s}\right\}$$
$$= \frac{4}{6}e^{2t}u(t) + \frac{1}{3}e^{-t}u(t) - u(t)$$

Problem 2: Consider the continuous time causal filter described by the differential equation
$$\frac{d^2y}{dt^2} + 1\frac{dy}{dt} + 4y = x$$
Compute the steady-state response of the filter to $x(t) = cos[2t + 1]u(t)$.

$$H(s) = \frac{1}{s^2 + s + 4}$$
The system is stable (transfer function poles or roots of the denominator are $(-1 \pm \sqrt{1-16})/2$ are in the LHP, i.e., have negative real parts), hence the steady-state response is well-defined.

$$y_{ss}(t) = |H(j2)|\cos(2t + 1 + \angle[H(j2)]) =$$
$$= \frac{1}{|(j2)^2 + j2 + 4|}\cos\left(2t + 1 - atan\frac{1}{0}\right) = \frac{1}{2}\cos\left(2t + 1 - atan\frac{1}{0}\right) =$$
$$= \frac{1}{2}\cos(2t - 32.7^o)$$

Problem 1:
Find the largest sampling interval T_s to allow perfect reconstruction of the signals:
(NOTE: h*x denotes convolution of h and x)

1. $\dfrac{\sin 4t}{4t} * \dfrac{\sin 5t}{3t}$

2. $\dfrac{\sin t}{t} + \dfrac{\sin 2t}{t^2}$

3. $\dfrac{\sin t}{t} * \sin 5t$

Using the shortcut method, we compute the Nyquist rates (2 x max signal frequency) of the individual signals and then estimate the Nyquist rate of the composite signal:

1. $w_{N1} = 8, w_{N2} = 10 \Rightarrow w_N = \min(w_{N1}, w_{N2}) = 8 \Rightarrow T_s = \dfrac{2\pi}{w_N} = \dfrac{\pi}{4}$

2. $w_{N1} = 2, w_{N2} = \infty \Rightarrow w_N = \max(w_{N1}, w_{N2}) = \infty \Rightarrow T_s = 0$

3. $w_{N1} = 2, w_{N2} = 10 \Rightarrow w_N = \min(w_{N1}, w_{N2}) = 2 \Rightarrow T_s = \dfrac{2\pi}{2} = \pi$

> Note: In fact, a direct computation of #3 shows that the resulting signal is 0, hence any sampling rate can be used. This is consistent with our understanding of the shortcut method producing conservative estimates of the sampling rate.

Problem 2:
For an ideal sampling process with sampling time 5ms, determine the cutoff frequency of the ideal low-pass filter needed to avoid any aliasing effects. Design an ideal low pass filter for reconstruction. What is the maximum frequency of signals that can be sampled and reconstructed with this system?

With a sampling time of 5ms, the sampling frequency is 200Hz so the maximum signal frequency that can be recovered after sampling and reconstruction is 100Hz. Then, the cutoff frequency of the ideal low-pass Anti-Aliasing Filter is 100Hz, (or less). The reconstruction filter should have amplitude T = 0.002 and cutoff frequency 100Hz. The maximum frequency of signals that can be reconstructed is also 100Hz.

Problem 1: Let $x(t)$ be the periodic signal shown in the figure below (sawtooth wave with offset and shift).
Compute the coefficients a_k of the Fourier series expansion of $x(t)$.
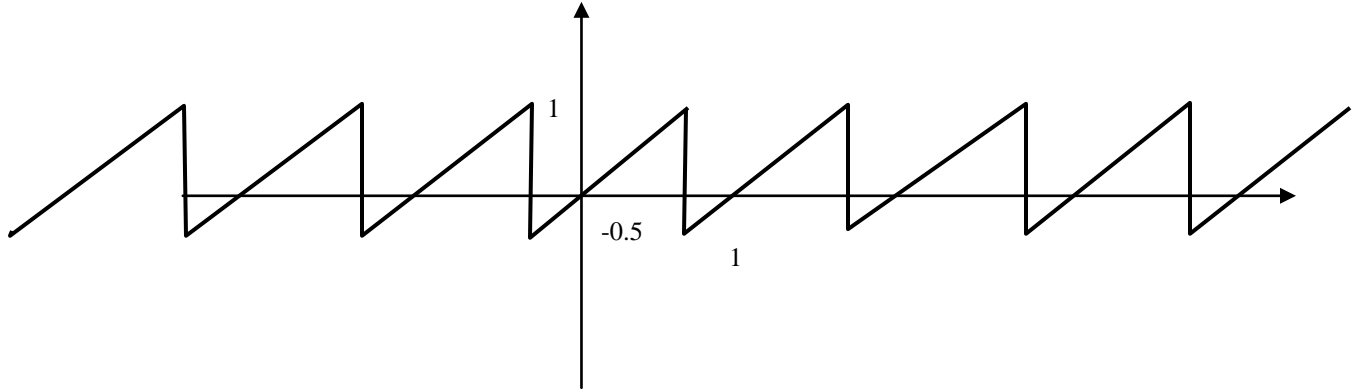
The derivative of x is $\frac{dx}{dt} = 1.5 - \sum_n 1.5\delta\left(t - n + \frac{1}{3}\right)$. From the tables, the Fourier series (FS) coefficients of the shifted impulse train are $a_k = -1.5(1)e^{jk2\pi/3}, \forall k$. Then, the FS coefficients of x, say b_k are given by $b_k = \frac{1}{jkw_0}a_k, \ k \neq 0, \ w_0 = \frac{2\pi}{1}$. Thus, $b_k = \frac{-1.5}{jk2\pi}e^{jk2\pi/3}, \ k \neq 0$.

For k = 0, we compute the FS coefficient directly from the definition, $b_0 = \frac{1}{T}\int_T x(t)dt = -\frac{1}{3} \times \frac{1}{2} \times \frac{1}{2} + \frac{2}{3} \times 1 \times \frac{1}{2} = \frac{3}{12} \Rightarrow b_0 = \frac{1}{4}$

Problem 2: Let $X(jw)$ be the Fourier transform of $x(t) = e^{-2\,|t+1|}$. Find $X(j0)$ and $\int_{-\infty}^{\infty} wX(jw)dw$.

From the definition of the Fourier transform and its inverse, $X(j0) = \int x(t)e^{-j0t}dt = \int e^{-2|t+1|}dt \Rightarrow$
$$X(j0) = 1.$$
On the other hand, $\frac{dx}{dt}(0) = \frac{1}{2\pi}\int (jwX(jw))e^{jw0}dw \Rightarrow \int wX(jw)dw = \frac{2\pi}{j}\frac{dx}{dt}(0) = \left(\frac{2\pi}{j}\right)(-2e^{-2}) \Rightarrow$
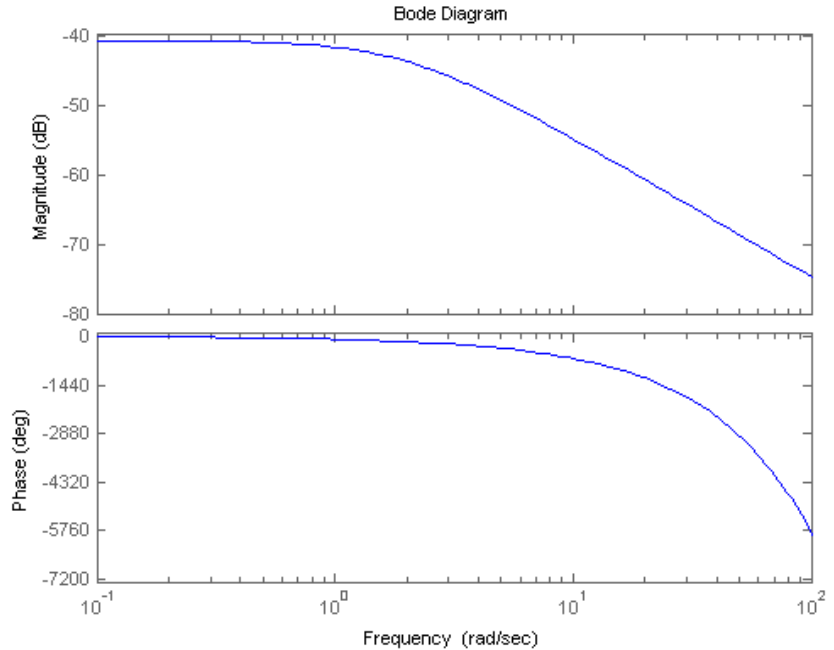$$\int wX(jw)dw = 4\pi je^{-2}$$

Problem 3:
Consider the filter with impulse response $h(t) = e^{-2(t+1)}u(t-1)$.
1. Find the transfer function and sketch the Bode Plot
2. Find the Fourier transform of the output when $x(t) = e^{-t}u(t)$ and when $x(t) = \sin(2t)$
3. Find the output when $x(t) = e^{-t}u(t)$ (a) using convolution, and (b) taking the inverse Fourier transform of your answer to part 2.

$$h(t) = e^{-2(t+1)}u(t-1) = e^{-4}e^{-2(t-1)}u(t-1) \Rightarrow H(jw) = \frac{e^{-4}e^{-jw}}{jw+2}$$

$$|H(jw)| = \frac{e^{-4}}{\sqrt{w^2+4}}, \qquad \angle H(jw) = -w - \tan^{-1}\frac{w}{2}$$

In MATLAB,
`>> H=tf(exp(-4),[1 2])`
`>> H.iodelay=1`
`>> bode(H)`

Bode Diagram

2.

Exponential input : $F\{x\} = \dfrac{1}{jw+1}$, $\quad Y(jw) = H(jw)X(jw) = \dfrac{e^{-4}e^{-jw}}{(jw+1)\ (jw+2)}$

Sinusoid input : $F\{x\} = F\{\sin 2t\} = \dfrac{\pi}{j}[\delta(w-2) - \delta(w+2)]$, $Y(jw) = H(jw)X(jw) =$

$$= \frac{\pi}{j}e^{-4}\left[\frac{e^{-2j}}{2j+2}\delta(w-2) - \frac{e^{2j}}{-2j+2}\delta(w+2)\right] \text{(evaluate H(jw) at the sin frequency)}$$

3. $y(t) = (h * x)(t) = e^{-4}\displaystyle\int_{-\infty}^{\infty} e^{-2(t-1-\tau)}u(t-1-\tau)e^{-\tau}u(\tau)d\tau = e^{-4}e^{-2(t-1)}u(t-1)\displaystyle\int_{0}^{t-1}e^{\tau}d\tau =$

$\qquad = e^{-4}[e^{-(t-1)} - e^{-2(t-1)}]u(t-1)$

$\qquad = F^{-1}\{Y(jw)\} = F^{-1}\left\{\dfrac{e^{-4}e^{-jw}}{(jw+1)\ (jw+2)}\right\} = e^{-4}F^{-1}\left\{\dfrac{1}{(jw+1)} + \dfrac{-1}{(jw+2)}\right\}\Bigg|_{t-1} =$

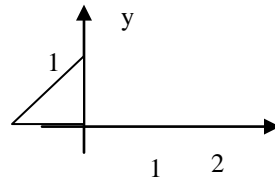$\qquad = e^{-4}[e^{-t}u(t)\big|_{t-1} - e^{-2t}u(t)\big|_{t-1}] = e^{-4}[e^{-(t-1)} - e^{-2(t-1)}]u(t-1)$

Problem 1. Compute the convolution $h*x$ when $x(t) = r(t)\text{-}r(t\text{-}1)\text{-}u(t\text{-}1)$, $h(t)=\delta(t+1)$.
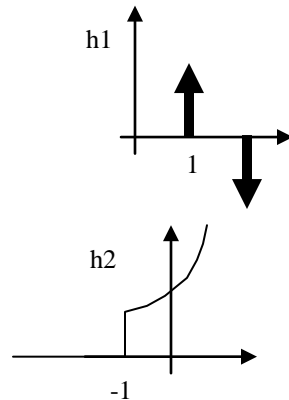
X

h

-1

1
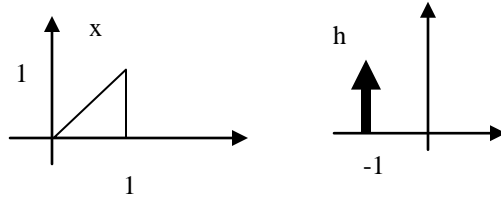
$(u(t)$ is the unit step, $r(t)$ is the unit ramp)

$(h * x)(t) = r(t + 1) - r(t) - u(t)$, or

y

1   2

Problem 2. Consider the filters

h1

1

1. $y(t) = x(t - 1) - x(t - 2)$
2. $y(t) = \int_{-\infty}^{t-1} e^{-\tau+t} x(\tau + 2) d\tau$
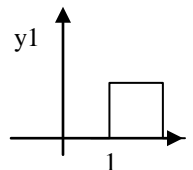
h2

1. Find and graph their impulse responses.
    1. $h(t) = \delta(t - 1) - \delta(t - 2)$
    2. $y(t) = \int_{-\infty}^{t-1} e^{-\tau+t} \delta(\tau + 2) d\tau = \int_{-\infty}^{t-1} e^{2+t} \delta(\tau + 2) d\tau = e^{2+t} u(t + 1)$

-1

2. Find and graph their step responses.
    1. $h(t) = u(t - 1) - u(t - 2)$
    2. $y(t) = \int_{-\infty}^{t-1} e^{-\tau+t} u(\tau + 2) d\tau = e^t u(t + 1) \int_{-2}^{t-1} e^{-\tau} d\tau = -e^t[e^{-t+1} - e^2]u(t + 1)$
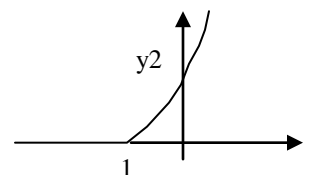 $= -[e^1 - e^{t+2}]u(t + 1)$

y1

3. Which filters are causal? (Justify)
    1. Causal because $h(t) = 0$, for t < 0.
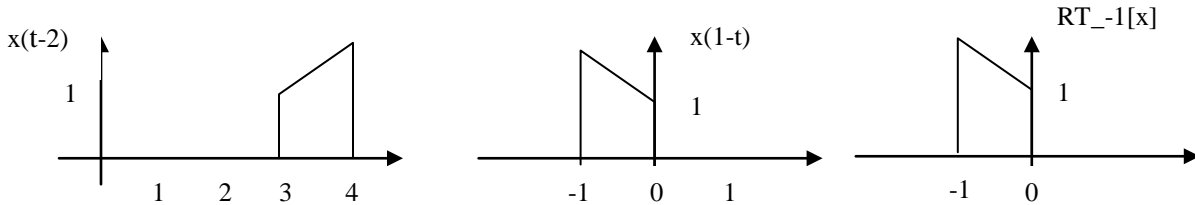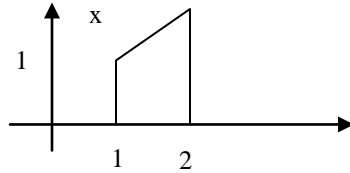   2. Non-Causal because $h(t) \neq 0$, for some t < 0.

1

4. Which filters are stable? (Justify)
    1. Stable because $\int |h| = 2 < \infty$
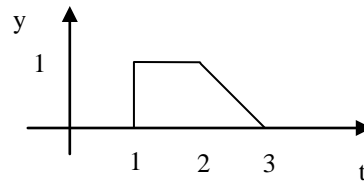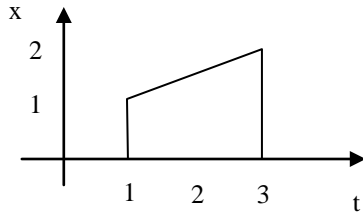   2. Unstable because $\int |h|$ diverges.

y2

1

Problem 1.

Consider the signal *x(t)* whose graph is shown below. Sketch the following signals:
x(t-2), x(1-t), RT-₁[x], where R denotes the reflection operation and T_{t0} denotes shift delay operation by t0.

Problem 2.

Describe the following signals in terms of elementary functions (δ, *u ,r*, …) and compute
$\int_{-\infty}^{\infty} x(t)\delta(t-1)dt$ and $\int_{-\infty}^{\infty} y(t)\delta(t-3)dt$.

$$x(t) = u(t-1) + \frac{1}{2}r(t-1) - \frac{1}{2}r(t-3) - 2u(t-3)$$

$$\int x(t)\delta(t-1)dt = \frac{[x(1^+) + x(1^-)]}{2} = \frac{1}{2}$$

$$y(t) = u(t-1) - r(t-2) + r(t-3)$$

$$\int y(t)\delta(t-3)dt = \frac{[y(3^+) + y(3^-)]}{2} = 0$$

EEE 203, Test 6   NAME:___SOLUTIONS_____

Problem 1: Consider the causal filter described by the difference equation

$$y[n+1] = \frac{1}{2} y[n] + \frac{1}{4} x[n-1]$$

1. Determine the transfer function

Taking z-transforms of both sides,

$$zY(z) = \frac{1}{2} Y(z) + \frac{1}{4} z^{-1} X(z) \Rightarrow H(z) \triangleq \frac{Y(z)}{X(z)} = \frac{\frac{1}{4}}{z\left(z - \frac{1}{2}\right)}$$

2. Compute the response of the filter to $x[n] = u[n]$

$$X(z) = \frac{z}{z-1}$$

$$Y(z) = \frac{\frac{1}{4}}{z\left(z - \frac{1}{2}\right)} \frac{z}{z-1} = \frac{\frac{1}{4}}{(z-1)\left(z - \frac{1}{2}\right)} = \frac{\frac{1}{2}}{(z-1)} + \frac{-\frac{1}{2}}{\left(z - \frac{1}{2}\right)}$$

$$y(n) = \frac{1}{2} u(n-1) - \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

Problem 2: Compute the steady-state response of the following discrete-time, causal filters to $x[n] = u[n-12]$:

$$H_1(z) = \frac{z+2}{(z)(z+0.5)}, \quad \text{Stable, } y_{ss}(n) = H_1\left(e^{j0}\right)e^{j0n} = H_1(1) = 2$$

$$H_2(z) = \frac{z-1}{(z+2)(z+0.8)}, \quad \text{Unstable, steady-state is not well-defined}$$

$$H_3(z) = \frac{z-0.1}{(z-0.5)(z-0.8)}, \quad \text{Stable, } y_{ss}(n) = H_1\left(e^{j0}\right)e^{j0n} = H_1(1) = 9$$

$$H_4(z) = \frac{z-10}{(z)(z+2)}, \quad \text{Unstable, steady-state is not well-defined}$$

 CLOSED BOOK & NOTES. TRANSFORM TABLES ALLOWED. 30'

Problem 1: Consider the continuous time causal filter with transfer function

$$H(s) = \frac{s}{(s+3)(s-1)}$$

Compute the response of the filter to x(t) = u(t) – u(t-1).

$y(t) = y_s(t) - y_s(t-1)$, where, $y_s(t) = L^{-1}\left\{H(s)\frac{1}{s}\right\}$, the step response.

$y_s(t) = L^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\} = L^{-1}\left\{\frac{-\frac{1}{4}}{(s+3)} + \frac{\frac{1}{4}}{(s-1)}\right\} = -\frac{1}{4}e^{-3t}u(t) + \frac{1}{4}e^{t}u(t) \Rightarrow$

$y(t) = -\frac{1}{4}e^{-3t}u(t) + \frac{1}{4}e^{t}u(t) - (-\frac{1}{4}e^{-3(t-1)}u(t-1) + \frac{1}{4}e^{t-1}u(t-1)$

Problem 2: Consider the continuous time causal filter described by the differential equation

$$2\frac{d^2y}{dt^2} + \frac{dy}{dt} + 8y = 2x + 4\frac{dx}{dt}$$

Compute the steady-state response of the filter to x(t) = sin(2t-3)u(t-3) + u(t-2).

$H(s) = \frac{4s+2}{2s^2+s+8}$. Its poles are $\frac{(-1\pm\sqrt{1-4\cdot2\cdot8})}{2\cdot2}$ and have negative real parts, so the system is stable and the steady-state response is well-defined.

The steady-state input is $\sin(2t-3) + 1$.
The steady-state output is $H(j2)\sin\big(2t-3-\angle H(j2)\big) + 1 \cdot H(j0) = 4.12\sin(2t-3-14^o) + \frac{1}{4}$
$$= 4.12\sin(2t-3.245) + 0.25$$

Problem 1:
Find the largest sampling interval T_s to allow perfect reconstruction of the signals:
(NOTE: h*x denotes convolution of h and x)

1. $\sin 2t * e^{-3t}u(t)$: $w_N = 2(\min(2, \infty)) = 4 \Rightarrow T_s = \frac{2\pi}{4} = \frac{\pi}{2}$.

2. $\frac{\sin(t)}{t} * \cos t$: $w_N = 2(\min(1,1)) = 2 \Rightarrow T_s = \frac{2\pi}{2} = \pi$

3. $\frac{\cos(t)}{t}\sin(t) = \frac{\sin(t)}{t}\cos(t)$: $w_N = 2(\text{sum}(1,1)) = 4 \Rightarrow T_s = \frac{2\pi}{4} = \frac{\pi}{2}$

Problem 2:
The frequency spectrum of a vibration signal is shown in the figure below. We would like to sample and analyze the spectral peaks around 3kHz but our computer can only support sampling rates up to 8kHz. Comment on the feasibility of this objective and describe the ideal components that should be used in such a sampling system.
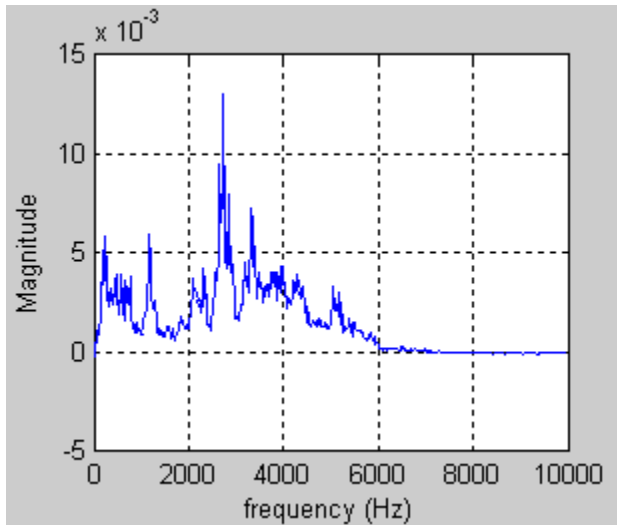
To analyze the spectral peaks around 3kHz we need to sample faster than 6kHz (plus any margin desirable) so the computer has sufficient capabilities, at least in principle. From the given spectrum, there are components in the signal with frequencies higher than Nyquist (4kHz) so we need to use a good Anti-Aliasing Filter with cutoff frequency below 4kHz, but above 3kHz to include the interesting portion of the signal. Then, if needed, reconstruct with a filter with cutoff higher than 3kHz to cover the signal, but below the 4kHz Nyquist frequency of the system. (The analysis may or may not require reconstruction.)

----→ AAF, 3-4kHz ------→ SAMPLING 6-8kHz ------→ Analysis --→ (if needed, Reconstruction Lowpass 3-4KHz, amplitude T = 1/3000-1/4000)

Problem 1: Consider the filter with impulse response $h(t) = e^{-t}u(t+1)$.

1. Find the Fourier transform of the output Y(jw) when $x(t) = e^{-3t}u(t)$

2. Find the time-domain expression for the output y(t) when $x(t) = e^{-3t}u(t)$

1. $e^{-t}u(t+1) = ee^{-(t+1)}u(t+1) \xrightarrow{F} H(jw) = e\dfrac{e^{jw}}{jw+1} \Rightarrow Y(jw) = H(jw)X(jw) = e\dfrac{e^{jw}}{(jw+1)(jw+3)}$

2.

$y(t) = F^{-1}\{Y(jw)\} = F^{-1}\left\{e\dfrac{e^{jw}}{(jw+1)(jw+3)}\right\} = eF^{-1}\left\{\dfrac{1}{(jw+1)(jw+3)}\right\}_{t=t+1} = eF^{-1}\left\{\dfrac{1/2}{(jw+1)}\right\}_{t=t+1} + eF^{-1}\left\{\dfrac{-1/2}{(jw+3)}\right\}_{t=t+1}$

$= e\left[\dfrac{1}{2}e^{-(t+1)}u(t+1) - \dfrac{1}{2}e^{-3(t+1)}u(t+1)\right]$

Problem 2: Consider the RC filter described by the differential equation $a\dfrac{dy}{dt} + y = x$, where a is an adjustable

parameter.
1. Find the magnitude and phase of the frequency response of the filter.
2. Compute the value of a so that steady-state response of the filter to x[t] = cos[10t]u(t) has amplitude 0.01.

1. $H(jw) = \dfrac{1}{(ajw+1)} \xrightarrow{yields} |H(jw)| = \dfrac{1}{\sqrt{a^2w^2+1}}$, $\angle H(jw) = -\tan^{-1}(aw)$

2. $y(t) = |H(jw)|\cos(wt + \angle H(jw)) \xrightarrow{yields} \left\{\dfrac{1}{\sqrt{a^2w^2+1}}\right\}_{w=10} = 0.01 \Rightarrow a \simeq 10$

Problem 3: Let *x(t)* be the periodic signal shown in the figure below (square wave with offset).
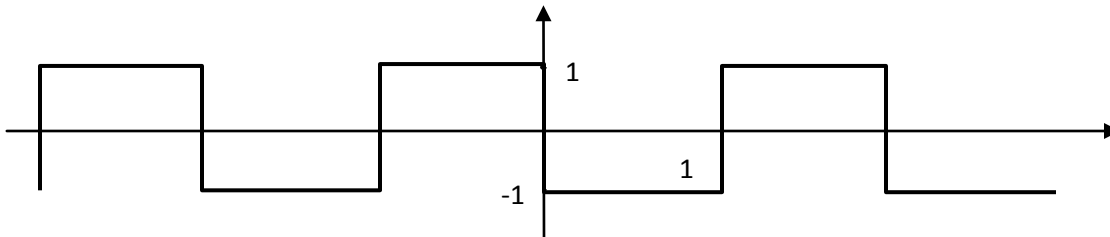
Compute the coefficients a_k of the Fourier series expansion of *x(t).*

Several possible approaches, e.g., use $\dfrac{dx}{dt} = \sum\delta(t-n\dots)$ (two delta trains), from which, $F\{x\} = \dfrac{1}{jkw_0}F\left\{\dfrac{dx}{dt}\right\}$, etc.

Here, we use $x_0(t) = standard\ square\ wave, T = 2, w_0 = \pi,$. Then $x(t) = 2x_0\left(t+\dfrac{1}{2}\right) - 1$.

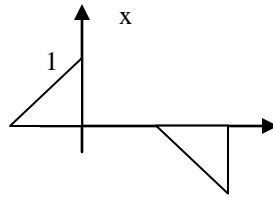Hence, $b_k = 2a_ke^{jk\pi/2} = \dfrac{2}{k\pi}\sin\left(\dfrac{k\pi}{2}\right)e^{jk\pi/2} = \dfrac{2j}{k\pi}\sin^2\left(\dfrac{k\pi}{2}\right), b_0 = \dfrac{1}{T}\int x\,dt = 0.$

EEE 203, Test 2   NAME:_____SOLUTIONS_____

Problem 1. Compute the convolution $h*x$ when $x(t) = r(t)-u(t-1)-r(t-2)-u(t-2)$, $h(t)=\delta(t+1)-\delta(t)$.
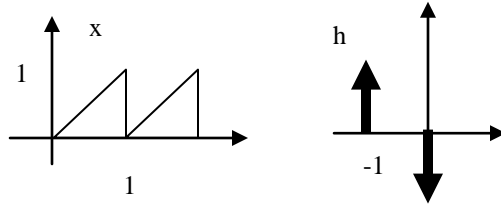
($u(t)$ is the unit step, $r(t)$ is the unit ramp)

$$h * x = x(t + 1) - x(t) = r(t + 1) - u(t) - r(t) - \{r(t - 1) - u(t - 2) - r(t - 2)\}$$

Problem 2. Consider the filters

1. $y(t) = x(t + 1) - 2x(t) + x(t - 1)$
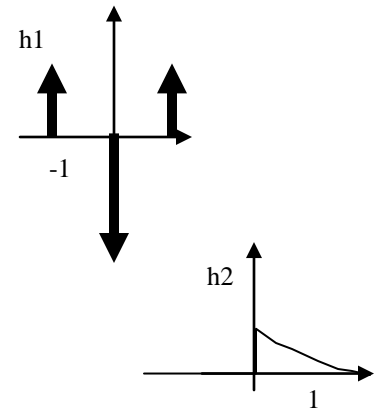2. $y(t) = \int_{-\infty}^{t-2} e^{\tau-t}x(\tau + 2)d\tau$

1. Find and graph their impulse responses.

2. Which filters are causal? (Justify)

3. Which filters are stable? (Justify)

1.1 $h(t) = \delta(t + 1) - 2\delta(t) + \delta(t - 1)$

2.1 This filter is not causal because h(t) is not zero for some t<0.
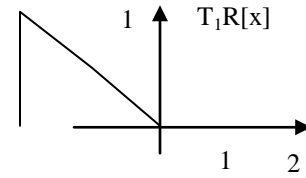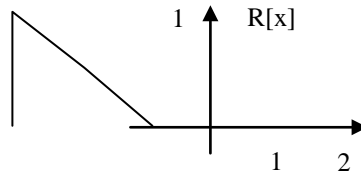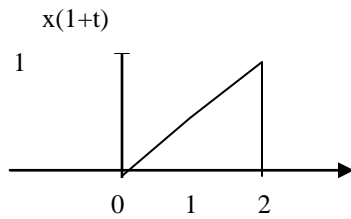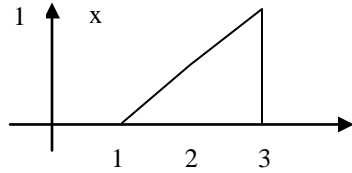3.1 This filter is stable because $\int |h| = 4 < \infty$

1.2 $h(t) = \int_{-\infty}^{t-2} e^{-(t-\tau)}\delta(\tau + 2)d\tau = e^{-(t+2)}u(t)$

2.2 This filter is causal because h(t) = 0 for all t<0.
3.2 This filter is stable because h(t) is absolutely integrable $\int |h| = e^{-2} < \infty$.
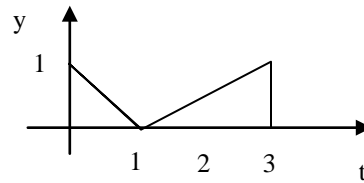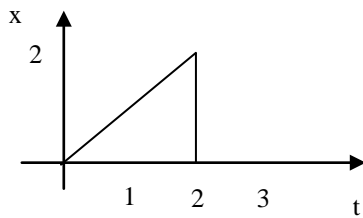
Problem 1.
Consider the signal *x(t)* whose graph is shown below. Sketch the following signals:
x(t+1), *T₁R[x]*, where R denotes the reflection operation and T_{t0} denotes shift delay operation by t0.

x(1+t)

R[x]

T₁R[x]

Problem 2.
Describe the following signals in terms of elementary functions (δ, u ,r, …) and compute $\int_{-\infty}^{\infty} x(t)\delta(t-1)dt$ and $\int_{-\infty}^{\infty} y(t)\delta(t-1)dt$.
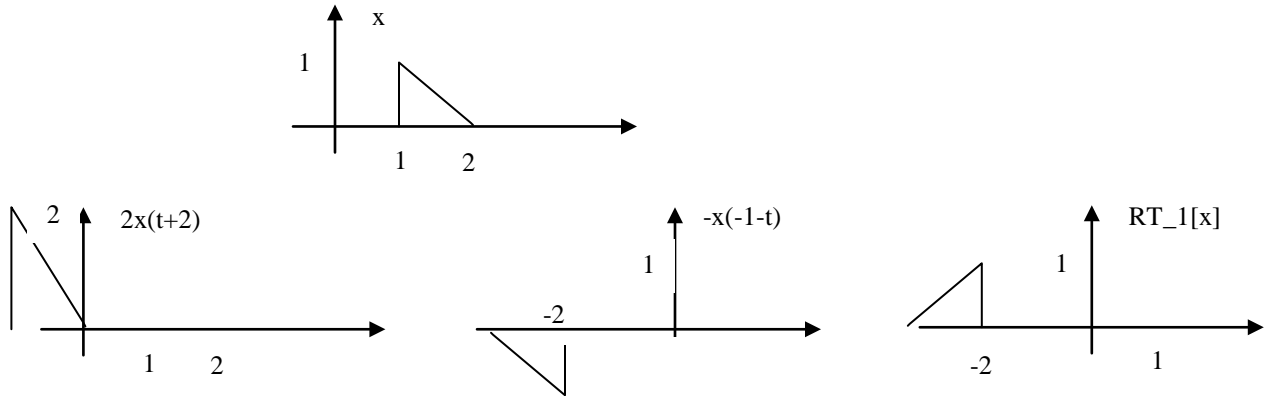
$$x(t) = r(t) - r(t-2) - 2u(t-2), \qquad \int x(t)\delta(t-1)dt = x(1) = 1$$
$$y(t) = u(t) - r(t) + \frac{3}{2}r(t-1) - \frac{1}{2}r(t-3) - u(t-3), \qquad \int y(t)\delta(t-1)dt = y(1) = 0$$
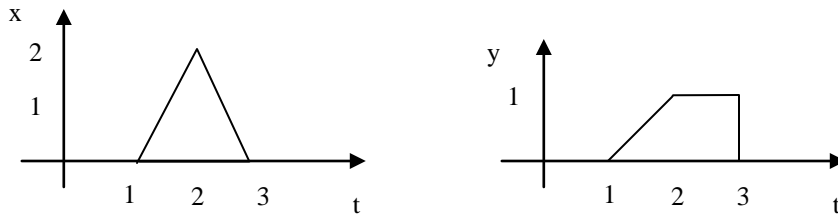
Problem 1.
Consider the signal *x(t)* whose graph is shown below. Sketch the following signals:
2x(t+2), -x(-1-t), RT₁[x], where R denotes the reflection operation and T_{t0} denotes shift delay operation by t0.

Problem 2.
Describe the following signals in terms of elementary functions (δ, *u* ,*r*, …) and compute
$\int_{-\infty}^{\infty} x(t)\delta(t-2)dt$ and $\int_{-\infty}^{\infty} y(t)\delta(t-3)dt$.

$$x(t) = 2r(t-1) - 4r(t-2) + 2r(t-3)$$
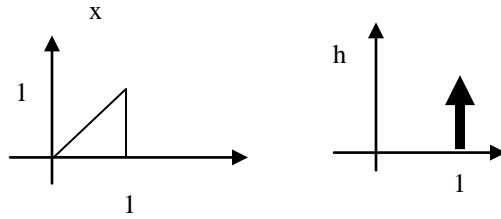$$\int x(t)\delta(t-2)dt = \frac{[x(2^+) + x(2^-)]}{2} = 2$$
$$y(t) = r(t-1) - r(t-2) - u(t-3)$$
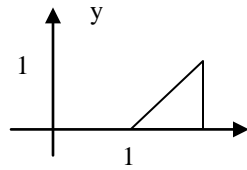$$\int y(t)\delta(t-3)dt = \frac{[y(3^+) + y(3^-)]}{2} = \frac{1}{2}$$

Problem 1. Compute the convolution $h*x$ when $x(t)$, $h(t)$ are as shown below.
Express both signals in terms of elementary functions and use both an analytical and a
"graphical" approach

$$(h * x)(t) = \delta(t-1) * [r(t) - r(t-1) - u(t-1)] = r(t-1) - r(t-2) - u(t-2)$$

Problem 2. Consider the filters

1. $y(t) = x(t+1) - x(t-2)$
2. $y(t) = \int_{-\infty}^{t-1} e^{-\tau+t} x(\tau+1) d\tau$

1. Find and graph their impulse responses.
1. $h_1(t) = \delta(t+1) - \delta(t-2)$
2. $h_2(t) = \int_{-\infty}^{t-1} e^{-\tau+t} \delta(\tau+1) d\tau = \int_{-\infty}^{t-1} e^{1+t} \delta(\tau+1) d\tau = e^{1+t} u(t+1-1) = e^{1+t} u(t)$

2. Find and graph their step responses.
1. $y_1(t) = u(t+1) - u(t-2)$
2. $y_2(t) = \int_{-\infty}^{t-1} e^{-\tau+t} u(\tau+1) d\tau = e^t \int_{-1}^{t-1} e^{-\tau} d\tau\, u(t-1+1)$
   $= -e^t [e^{-(t-1)} - e^1] u(t) = [e^{t+1} - e^1] u(t)$

3. Which filters are causal? (Justify)
(1) is not causal because $h_1(t) \neq 0$ for some $t < 0$
(2) is causal because $h_2(t) = 0$ for all $t < 0$

4. Which filters are stable? (Justify)
(1) is stable because $\int |h_1(t)| dt = 2 < \infty$
(2) is not stable because $\int |h_2(t)| dt\ diverges$
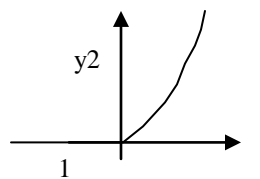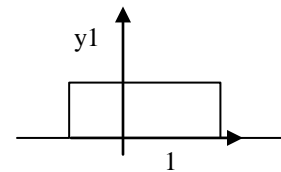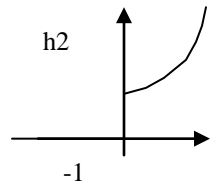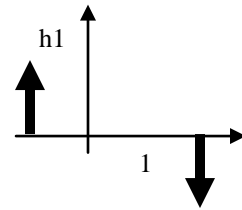
Problem 1: Let $x(t)$ be the periodic signal shown in the figure below (sawtooth wave with offset and shift).
Compute the coefficients a_k of the Fourier series expansion of $x(t)$.

The derivative of x is $\frac{dx}{dt} = \frac{3}{2} - \sum_n (3)\delta(t - n + 1)$. From the tables, the Fourier series (FS) coefficients of the shifted impulse train are $a_k = -3(1/2)e^{jk2\pi/2}, \forall k$. Then, the FS coefficients of x, say b_k are given by $b_k = \frac{1}{jkw_0} a_k$, $k \neq 0$, $w_0 = \frac{2\pi}{2}$. Thus, $b_k = \frac{-3}{jk2\pi} e^{jk\pi}, \ k \neq 0$.

For k = 0, we compute the FS coefficient directly from the definition, $b_0 = \frac{1}{T}\int_T x(t)dt =$
$\frac{1}{2}\int_T \frac{3}{2}t + \frac{1}{2}dt = \frac{1}{2}$
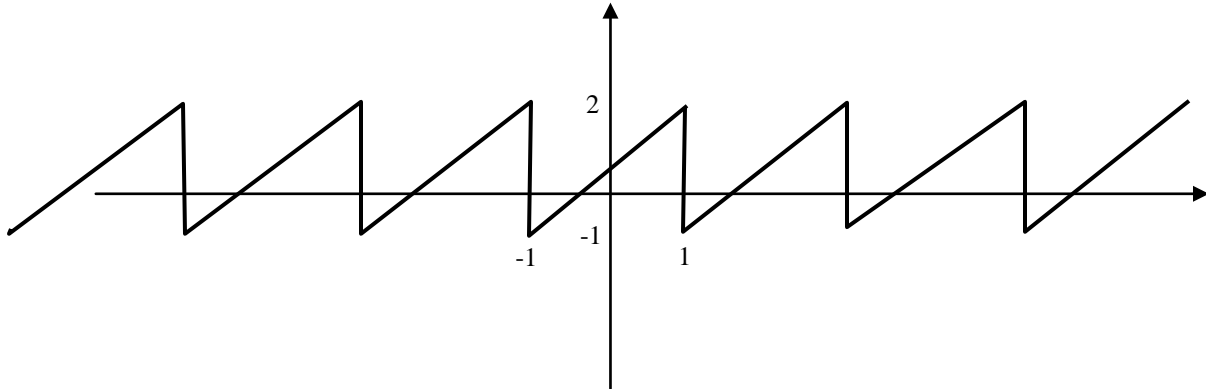
Problem 2: Let $X(jw)$ be the Fourier transform of $x(t) = e^{-2|t|}$. Find $X(j0)$ and $\int\limits_{-\infty}^{\infty} jwX(jw)dw$.

From the definition of the Fourier transform and its inverse, $X(j0) = \int x(t)e^{-j0t}dt = \int e^{-2|t|}dt \Rightarrow$

$X(j0) = 1$.

On the other hand, $\frac{dx}{dt}(0) = \frac{1}{2\pi}\int (jwX(jw))e^{jw0}dw \Rightarrow \int wX(jw)dw = \frac{2\pi}{j}\frac{dx}{dt}(0) =$
$\left(\frac{2\pi}{j}\right)(2 - 2)/2 \ (average \ of \ left \ and \ right \ limits) \Rightarrow$

$\int wX(jw)dw = 0$

Problem 3:
Consider the filter with impulse response $h(t) = e^{-(t+1)}u(t+1)$.
1. Find the transfer function and sketch the Bode Plot
2. Find the Fourier transform of the output when $x(t) = e^{-2t}u(t)$ and when $x(t) = \sin(2t)$
3. Find the output when $x(t) = e^{-2t}u(t)$ (a) using convolution, and (b) taking the inverse Fourier transform of your answer to part 2.

$$h(t) = e^{-(t+1)}u(t+1) \Rightarrow H(jw) = \frac{e^{jw}}{jw+1}$$

$$|H(jw)| = \frac{1}{\sqrt{w^2+1}}, \qquad \angle H(jw) = w - \tan^{-1}w$$

In MATLAB,
H=tf(1,[1 1])
%Matlab does not accept
% -ve delays
[m,p,w]=bode(H);
m=m(:);p=p(:);w=w(:);
figure(1);subplot(211),loglog(w,m),title('magnitude')
subplot(212),loglog(w,m),semilogx(w,p+w*180/pi),title('phase')
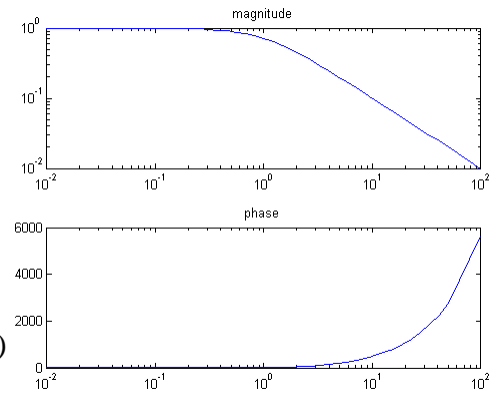
2.

Exponential input : $F\{x\} = \dfrac{1}{jw+2}$, $\quad Y(jw) = H(jw)X(jw) = \dfrac{e^{jw}}{(jw+1)\ (jw+2)}$

Sinusoid input : $F\{x\} = F\{\sin 2t\} = \dfrac{\pi}{j}[\delta(w-2) - \delta(w+2)]$, $Y(jw) = H(jw)X(jw) =$

$$= \frac{\pi}{j}\left[\frac{e^{2j}}{2j+1}\delta(w-2) - \frac{e^{-2j}}{-2j+1}\delta(w+2)\right] \text{(evaluate H(jw) at the sin frequency)}$$

3. $y(t) = (h*x)(t) = \displaystyle\int_{-\infty}^{\infty} e^{-(t-\tau+1)}u(t+1-\tau)e^{-2\tau}u(\tau)d\tau = e^{-(t+1)}u(t+1)\int_{0}^{t+1}e^{-\tau}d\tau =$

$$= [e^{-(t+1)} - e^{-2(t+1)}]u(t+1)$$

$$= F^{-1}\{Y(jw)\} = F^{-1}\left\{\frac{e^{+jw}}{(jw+1)\ (jw+2)}\right\} = F^{-1}\left\{\frac{1}{(jw+1)} + \frac{-1}{(jw+2)}\right\}\Bigg|_{t+1} =$$

$$= [e^{-t}u(t)\big|_{t+1} - e^{-2t}u(t)\big|_{t+1}] = [e^{-(t+1)} - e^{-2(t+1)}]u(t+1)$$

Problem 1:
Find the largest sampling interval T_s to allow perfect reconstruction of the signals:
(NOTE: h*x denotes convolution of h and x)

1. $\dfrac{\sin 4t}{4t} \cdot \dfrac{\sin 5t}{3t}$

2. $\dfrac{\sin^2 t}{t^2} + \dfrac{\sin 2t}{2t}$

3. $\dfrac{\sin 2t}{t} * \sin 2t$

Using the shortcut method, we compute the Nyquist rates (2 x max signal frequency) of the individual signals and then estimate the Nyquist rate of the composite signal:

1. $w_{N1} = 8, w_{N2} = 10 \Rightarrow w_N = (w_{N1} + w_{N2}) = 18 \Rightarrow T_s = \dfrac{2\pi}{w_N} = \dfrac{\pi}{9}$

2. $w_{N1} = 2 \times (1 + 1), w_{N2} = 4 \Rightarrow w_N = \max(w_{N1}, w_{N2}) = 4 \Rightarrow T_s = \dfrac{\pi}{2}$

3. $w_{N1} = 4, w_{N2} = 4 \Rightarrow w_N = \min(w_{N1}, w_{N2}) = 4 \Rightarrow T_s = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

Problem 2:
A sampling system is to be designed to handle signals with maximum frequency of interest 2kHz. Using ideal components, design the sampling time, determine the cutoff frequency of the anti-aliasing filter and design the reconstruction filter.

The Nyquist rate for 2kHz signals is 4kHz. Hence the sampling time should be less than 0.25ms, Then, the cutoff frequency of the ideal low-pass Anti-Aliasing Filter is 2kHz. The reconstruction filter should have amplitude T = 0.00025 and cutoff frequency 2kHz.

EEE 203, HW 5   NAME:___SOLUTIONS____

Problem 1: Consider the continuous time causal filter with transfer function
$$H(s) = \frac{s-1}{(s)(s+1)}$$
Compute the response of the filter to $x(t) = u(t)$

$$y(t) = L^{-1}\left\{\frac{s-1}{(s)(s+1)s}\right\} = L^{-1}\left\{\frac{-2}{(s+1)} + \frac{-1}{s^2} + \frac{2}{s}\right\} = -2e^{-t}u(t) - tu(t) + 2u(t)$$

Problem 2: Consider the continuous time causal filter described by the differential equation
$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 8y = \frac{dx}{dt}$$
Compute the steady-state response of the filter to $x(t) = cos[2t + 1]u(t)$.

$$H(s) = \frac{s}{s^2 + 2s + 8}$$
The system is stable (transfer function poles or roots of the denominator are $(-1 \pm j2.646)$ are in the LHP, i.e., have negative real parts), hence the steady-state response is well-defined.

$$y_{ss}(t) = |H(j2)|\cos(2t + 1 + \angle[H(j2)]) =$$
$$= 0.354\cos(2t + 1 + 45^o) = 0.354\cos(2t + 102.3^o) =$$

Problem 1: Consider the causal filter described by the difference equation

$$y[n] = \frac{1}{3} y[n-1] + \frac{1}{4} x[n-1]$$

1. Determine the transfer function
2. Compute the response of the filter to $x[n] = u[n]$
3. Compute the response to $x[n] = \delta[n]$
4. Compute the steady state response to $x[n] = \sin(^{n\pi}/_{16}) \, u[n-1]$

Note: For a stable system if x converges to a periodic signal x_s, then y converges to a periodic signal that is the system response to the x_s. This can be computed using Fourier theory. A useful simplification is:

For Continuous Time:
$$x(t) = e^{j\omega_o t} \Rightarrow y(t) = H(j\omega_o)e^{j\omega_o t}$$
$$x(t) = \cos(\omega_o t) \Rightarrow y(t) = |H(j\omega_o)|\cos(\omega_o t + \angle H(j\omega_o))$$

For Discrete Time:
$$x(n) = e^{j\Omega_o n} \Rightarrow y(n) = H(e^{j\Omega_o})e^{j\Omega_o n}$$
$$x(n) = \cos(\Omega_o n) \Rightarrow y(n) = |H(e^{j\Omega_o})|\cos(\Omega_o n + \angle H(e^{j\Omega_o}))$$

where, $H(s), H(z)$ are the continuous and discrete system transfer functions, respectively.

1. $H(z) = \frac{\frac{1}{4}z^{-1}}{1-\frac{1}{3}z^{-1}} = \frac{\frac{1}{4}}{z-\frac{1}{3}}$

2. $y(z) = H(z)\frac{z}{z-1} = \frac{\frac{1}{4}z}{\left(z-\frac{1}{3}\right)(z-1)} = z\left(\frac{-\frac{3}{8}}{\left(z-\frac{1}{3}\right)} + \frac{\frac{3}{8}}{(z-1)}\right) \Rightarrow y(n) = Z^{-1}\left\{\frac{-\frac{3}{8}z}{\left(z-\frac{1}{3}\right)} + \frac{\frac{3}{8}z}{(z-1)}\right\} =$

$$y(n) = -\frac{3}{8}\left(\frac{1}{3}\right)^n u(n) + \frac{3}{8}u(n) \Rightarrow \begin{bmatrix} n \\ y_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0.25 & 0.333 \end{bmatrix} \cdots$$

3. $y(z) = H(z)(1) = \frac{\frac{1}{4}}{\left(z-\frac{1}{3}\right)} \Rightarrow y(n) = Z^{-1}\left\{\frac{\frac{1}{4}z}{\left(z-\frac{1}{3}\right)}\right\}_{n \leftarrow n-1} =$

$$y(n) = \frac{1}{4}\left(\frac{1}{3}\right)^{n-1}u(n-1) \Rightarrow \begin{bmatrix} n \\ y_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0.25 & 0.083 \end{bmatrix} \cdots$$

4. The filter is stable since the pole (1/3) has magnitude less than one. Hence, the steady-state response is well-defined and $y_{ss}(n) = |H(e^{j\Omega})|\sin\left(\Omega n + \angle H(e^{j\Omega})\right)$, $\Omega = \frac{\pi}{16}$

Then, $y_{ss}(n) = \frac{\frac{1}{4}}{\sqrt{\left\{\left[\cos\left(\frac{\pi}{16}\right)-\frac{1}{3}\right]^2 + \left[\sin\left(\frac{\pi}{16}\right)\right]^2\right\}}}\sin\left(\frac{\pi}{16}n - \text{atan}\frac{\sin\left(\frac{\pi}{16}\right)}{\cos\left(\frac{\pi}{16}\right)-\frac{1}{3}}\right) = 0.370\sin\left(\frac{\pi}{16}n - 16.8^o\right)$

(Verify in MATLAB: >> H=tf(1/4,[1 -1/3],1), [m,p]=bode(H,pi/16))

Problem 2: Consider the discrete time causal filters with transfer functions

$$H_1(z) = \frac{z-1}{(z-0.3)(z+0.4)}, \quad H_2(z) = \frac{z-1}{(z+0.5)(z-2)}$$

1. Determine their stability properties.
2. Compute the response of the filters to $x[n] = u[n]$

1. The poles of H1 are 0.3, -0.4, both are less than 1 in magnitude (inside the unit circle), so the system is stable. The poles of H2 are -0.5, 2, the second has magnitude greater than 1 so the system is unstable.

2. $y_1(z) = H_1(z)\frac{z}{z-1} = z\left(\frac{1.43}{(z-0.3)} + \frac{-1.43}{(z+0.4)}\right) \Rightarrow y_1(n) = Z^{-1}\left\{\frac{1.43z}{(z-0.3)} + \frac{-1.43z}{(z+0.4)}\right\} = 1.43(0.3)^n u(n) -$
$1.43(-0.4)^n u(n) = [0, 1, -0.067, 0.138, -0.0181, \dots]$

2. $y_2(z) = H_2(z)\frac{z}{z-1} = \frac{z}{(z+0.5)(z-2)} = z\left(\frac{-0.4}{(z+0.5)} + \frac{0.4}{(z-2)}\right) \Rightarrow y_2(n) = Z^{-1}\left\{\frac{-0.4z}{(z+0.5)} + \frac{0.4z}{(z-2)}\right\} =$
$-0.4(-0.5)^n u(n) + 0.4(2)^n u(n) = [0, 1, 1.5, 3.25, 6.375, \dots]$

EEE 203, Test 1   NAME:___SOLUTIONS___

Problem 1.
Consider the signal *x(t)* whose graph is shown below. Sketch the following signals:
-x(-t+1), R*T-₁R[x]*, where R denotes the reflection operation and T$_{t0}$ denotes shift delay
operation by t0.
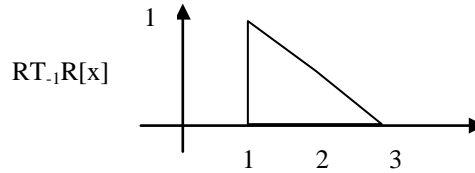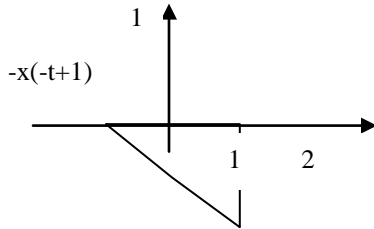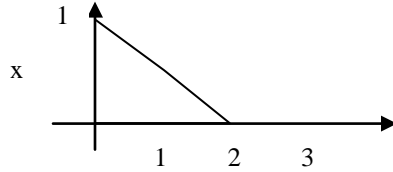
Problem 2.
Describe the following signals in terms of elementary functions (δ, *u* ,*r*, …) and compute
$\int_{-\infty}^{\infty} x(t)\delta(t-2)dt$ and $\int_{-\infty}^{\infty} y(t)\delta(t+3)dt$.
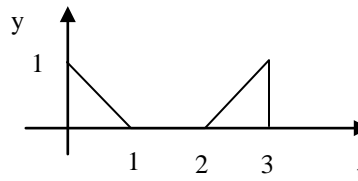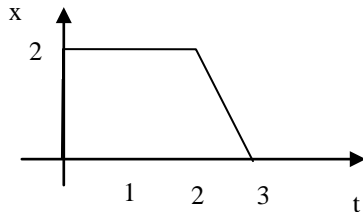
$$x(t) = 2u(t) - 2r(t-2) + 2r(t-3), \quad \int x(t)\delta(t-2)dt = x(2) = 2$$
$$y(t) = u(t) - r(t) + r(t-1) + r(t-2) - r(t-3) - u(t-3), \quad \int y(t)\delta(t+3)dt = y(-3) = 0$$

Problem 1. Compute the convolution $h*x$ when $h(t) = u(t+1), x(t) = \delta(t-1) - \delta(t-2)$

$$y(t) = h(t) * x(t) = u(t+1) * \delta(t-1) - u(t+1) * \delta(t-2) = u(t) - u(t-1)$$

Problem 2.
Consider the filters:

A. $y(t) = \int_t^\infty x(\tau)d\tau$

B. $y(t) = \int_{-\infty}^\infty e^{-(t-\tau)}u(t-\tau)x(\tau-1)d\tau$

1. Find and graph their impulse responses.

A. $h(t) = \int_t \delta(\tau)d\tau = u(-t)$

B. $h(t) = \int e^{-(t-\tau)}u(t-\tau)\delta(\tau-1)d\tau = \int e^{-(t-1)}u(t-1)\delta(\tau-1)d\tau = e^{-(t-1)}u(t-1)$

2. Which filters are causal? (Justify)

A is not causal, h(t) = u(-t) is not zero for all t < 0.

A is causal, h(t) = h(t)u(t-1) is zero for all t < 0.

3. Which filters are stable? (Justify)

A is not stable because h is not absolutely integrable

B is stable, because h is absolutely integrable $\int e^{-(t-1)}u(t-1)dt = -ee^{-t}|_1^\infty = 1$

Problem 1: Consider the filter with impulse response $h(t) = e^{-t}u(t-1)$.

1. Find the Fourier transform of the output Y(jw) when $x(t) = e^{-2t}u(t+1)$

2. Find the time-domain expression for the output y(t) when $x(t) = e^{-2t}u(t+1)$

$$Y(jw) = H(jw)X(jw) = \left[e^{-1}\frac{e^{-jw}}{jw+1}\right]\left[e^2\frac{e^{+jw}}{jw+2}\right] = e^1\frac{1}{(jw+2)(jw+1)}$$

$$y(t) = F^{-1}\{Y(jw)\} = F^{-1}\left\{e^1\frac{1}{(jw+2)(jw+1)}\right\} = F^{-1}\left\{e^1\frac{1}{(jw+1)}\right\} + F^{-1}\left\{e^1\frac{-1}{(jw+2)}\right\}$$
$$= e^1 e^{-t}u(t) - e^1 e^{-2t}u(t)$$

Problem 2: Consider the RC filter described by the differential equation $10\dfrac{dy}{dt} + y = x$

Write an expression for the magnitude and phase of the frequency response of the filter $Y(jw)/X(jw)$.

$$H(jw) = \frac{1}{10jw+1} \implies |H(jw)| = \frac{1}{\sqrt{100w^2+1}}, \quad \angle H(jw) = -\text{atan}(10w)$$

Problem 3: Let *x(t)* be the periodic signal shown in the figure below (square wave with offset).
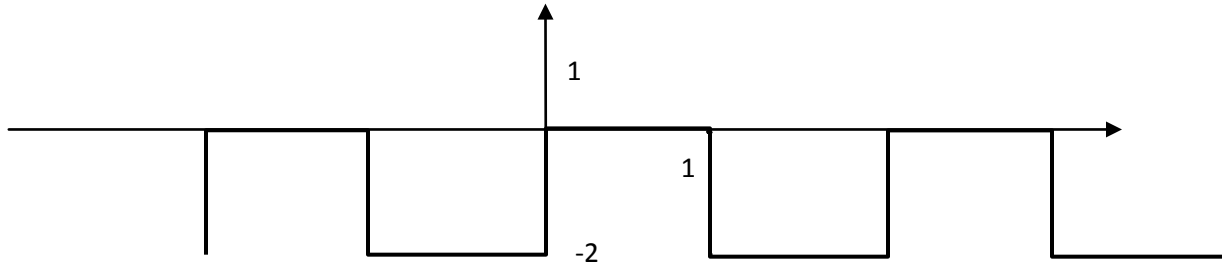
Compute the coefficients a_k of the Fourier series expansion of *x(t)*.

We observe that $x(t) = 2x_o(t-0.5) - 2$, where x_o is the standard square wave in the tables with period T = 2, $T_1 = \frac{1}{2}$. From the tables, the FS expansion for x_o is

$$\frac{\sin kw_0 T_1}{k\pi} = \frac{\sin\frac{k\pi}{2}}{k\pi}$$

Hence,

$$a_k = FS\{x\} = 2\frac{\sin\frac{k\pi}{2}}{k\pi}e^{-\frac{jk\pi}{2}}, \quad k \neq 0, \qquad a_0 = \frac{1}{T}\int_{<T>} x(t)dt = \frac{1}{2}(-2)(1) = -1$$

For the first term we may also observe that $e^{-\frac{jk\pi}{2}} = \cos\frac{k\pi}{2} - j\sin\frac{k\pi}{2}$ and the product of $\cos\frac{k\pi}{2}\sin\frac{k\pi}{2}$ is always zero. So,

$$a_k = -j2\frac{\sin^2\frac{k\pi}{2}}{k\pi}, \quad k \neq 0, \qquad a_0 = -1$$

CLOSED BOOK & NOTES. TRANSFORM TABLES ALLOWED. 30'

Problem 1:
Find the largest sampling interval T_s to allow perfect reconstruction of the signals:
(NOTE: h*x denotes convolution of h and x)

1. $\sin(5t + 10) + e^{-t}u(t)$ $w_{NYQ1} = 2 \times 5 = 10 \left(\frac{rad}{s}\right)$, $w_{NYQ2} = \infty$
=> $w_{NYQ} = \max \ (w_{NYQ1}, w_{NYQ2}) = \infty => T_s = 0$.

2. $\frac{\sin 5t}{t} * \sin t$ $w_{NYQ1} = 2 \times 5 = 10$, $w_{NYQ2} = 2 \times 1 = 2 \left(\frac{rad}{s}\right)$

$=> w_{NYQ} = \min \ (w_{NYQ1}, w_{NYQ2}) = 2 => T_s = \frac{2\pi}{2} = \pi$

3. $\cos^2(3t)\sin^2(2t)$ $w_{NYQ1} = 2 \times (3 + 3) = 12 \left(\frac{rad}{s}\right)$, $w_{NYQ2} = 2 \times (2 + 2) = 8$

$=> w_{NYQ} = \text{sum} \ (w_{NYQ1}, w_{NYQ2}) = 20 => T_s = \frac{2\pi}{20} = \frac{\pi}{10}$

Problem 2:
The frequency spectrum of a vibration signal is shown in the figure below. We would like to sample and analyze the spectral peaks around 3kHz but our sampling system does not include an anti-aliasing filter. Determine the sampling rate required for this task.
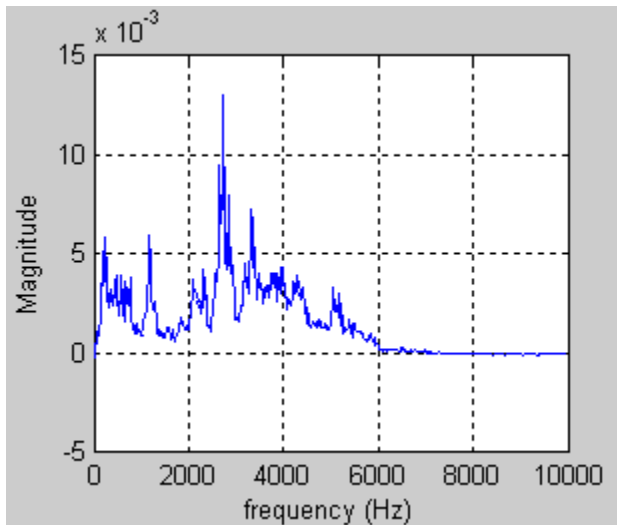
Since an anti-aliasing filter is not available, we need to sample faster than twice the largest frequency in the signal to make sure that there is no aliasing that would compromise the results of our analysis, regardless of the frequency of interest. From the plot, we estimate the highest frequency as 7kHz (6kHz or 8kHz are also valid estimates). Thus, the sampling rate required for the task is $f_{NYQ1} = 2 \times 7kHz = 14kHz$, $(T_s = 71\mu s)$.

  CLOSED BOOK & NOTES. TRANSFORM TABLES ALLOWED. 30'

Problem 1: Consider the continuous time causal filter with transfer function

$$H(s) = \frac{1}{(s+1)(s-1)}$$

Compute the response of the filter to x(t) = u(t) – u(t-1).

$y(t) = y_s(t) - y_s(t-1)$, where, $y_s(t) = L^{-1}\left\{H(s)\frac{1}{s}\right\}$, the step response.

$y_s(t) = L^{-1}\left\{\frac{1}{(s+1)(s-1)}\right\} = L^{-1}\left\{\frac{-1/2}{(s+1)} + \frac{1/2}{(s-1)}\right\} = -\frac{1}{2}e^{-1t}u(t) + \frac{1}{2}e^t u(t) \Rightarrow$

$y(t) = -\frac{1}{2}e^{-1t}u(t) + \frac{1}{2}e^t u(t) + \frac{1}{2}e^{-1(t-1)}u(t-1) - \frac{1}{2}e^{t-1}u(t-1)$

Problem 2: Consider the continuous time causal filter described by the differential equation

$$2\frac{d^2 y}{dt^2} + 8\frac{dy}{dt} + 1y = 2x - 4\frac{dx}{dt}$$

Compute the steady-state response of the filter to x(t) = sin(2t)u(t) + u(t-2).

$H(s) = \frac{-4s+2}{2s^2+8s+1}$. Its poles are($-0.13, -3.8$) and have negative real parts, so the system is stable and the steady-state response is well-defined.

The steady-state input is $\sin(2t) + 1$.
The steady-state output is $|H(j2)|\sin(2t + \angle H(j2)) + 1 \cdot H(j0) = 0.47\sin(2t - 190^o) + 2$

Note:

$$H(s) = \frac{-4s+2}{2s^2+8s+1} = \frac{-2s+1}{(s+0.13)(s+3.8)}$$

$$H(jw) = \frac{\sqrt{4w^2 + 1^2}}{\sqrt{w^2 + 0.13^2}\sqrt{w^2 + 3.8^2}} \angle \text{atan}(-2w) - \text{atan}\left(\frac{w}{0.13}\right) - \text{atan}\left(\frac{w}{3.8}\right)$$

EEE 203, Test 6   NAME:_____SOLUTIONS_____

Problem 1: Consider the causal filter described by the difference equation
$y[n+1] = -0.3y[n] + 0.3x[n-1]$
1. Determine the transfer function
Taking z-transforms of both sides,

$$zY(z) = -0.3Y(z) + 0.3z^{-1}X(z) \Rightarrow H(z) \triangleq \frac{Y(z)}{X(z)} = \frac{0.3}{z(z+0.3)}$$

2. Compute the response of the filter to $x[n] = u[n]$

$$X(z) = \frac{z}{z-1}, Y(z) = \frac{0.3}{z(z+0.3)}\frac{z}{z-1} = \frac{0.3}{(z-1)(z+0.3)} = \frac{\frac{0.3}{1.3}}{(z-1)} + \frac{-\frac{0.3}{1.3}}{(z+0.3)}$$

$$y(n) = \frac{0.3}{1.3}u(n-1) - \frac{0.3}{1.3}(-0.3)^{n-1}u(n-1)$$

Problem 2: Compute the steady-state response of the following discrete-time, causal filters to $x[n] = u[n-12]$:

$$H_1(z) = \frac{z-1}{(z)(z-0.5)}, \quad \text{Stable}, y_{ss}(n) = H_1(e^{j0})e^{j0n} = H_1(1) = 0$$

$$H_2(z) = \frac{z-1}{(z+2)(z-0.5)}, \quad \text{Unstable, steady-state is not well-defined}$$

Problem 3: Compute the steady-state response of the following discrete-time, causal filters to $x[n] = \cos\left(\frac{2\pi}{15}n\right)u[n-15]$:

$$H_3(z) = \frac{z-0.1}{(z+0.1)(z-0.5)}$$

The filter is stable since the poles have magnitude less than one. Hence, the steady-state response is well-defined and $y_{ss}(n) = |H(e^{j\Omega})|\cos\left(\Omega n + \angle H(e^{j\Omega})\right), \Omega = \frac{2\pi}{15}$

Then, $y_{ss}(n) = 1.44\sin\left(\frac{2\pi}{15}n - 39.8^o\right)$

$$|H(e^{j\Omega})| = \frac{\sqrt{\{[\cos(\frac{2\pi}{15})-0.1]^2+[\sin(\frac{2\pi}{15})]^2\}}}{\sqrt{\{[\cos(\frac{2\pi}{15})+0.1]^2+[\sin(\frac{2\pi}{15})]^2\}}\sqrt{\{[\cos(\frac{2\pi}{15})-0.5]^2+[\sin(\frac{2\pi}{15})]^2\}}} = 1.44$$

$$\angle H(e^{j\Omega}) = \text{atan}\frac{\sin(\frac{2\pi}{15})}{\cos(\frac{2\pi}{15})-0.1} - \text{atan}\frac{\sin(\frac{2\pi}{15})}{\cos(\frac{2\pi}{15})+0.1} - \text{atan}\frac{\sin(\frac{2\pi}{15})}{\cos(\frac{2\pi}{15})-0.5} = 39.8^o \text{ (all real parts positive)}$$