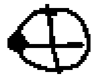
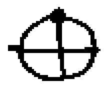
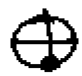
Ch. 1. Selected HW Problems

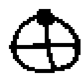
#1.1

1) $\frac{1}{2}e^{j\pi}$:  $\rightarrow \left(-\frac{1}{2}, 0\right) = \frac{1}{2}\left(\cos\pi + j\sin\pi\right)$

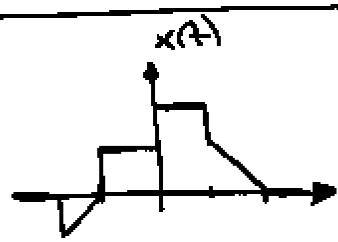
2) $\frac{1}{2}e^{-j\pi} = \frac{1}{2}\left[\cos(-\pi) + j\sin(-\pi)\right] = \left(-\frac{1}{2}, 0\right)$

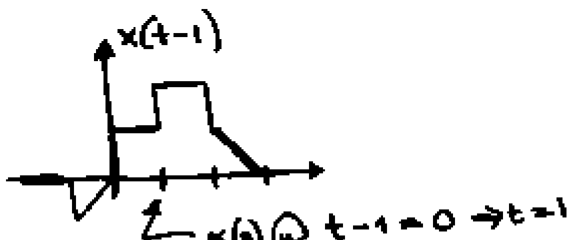
3) $e^{j\pi/2} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = (0,1)$ 

4) $e^{-j\pi/2} = (0,-1) = 0-j$ 

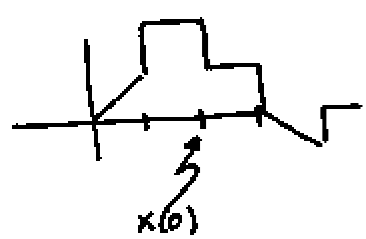
5) $e^{j5\pi/2} = (0,1) = 0+j$ 

#1.21





$x(2-t)$: $\begin{cases} \text{reverse direction} \\ x(0) \text{ @ } 2-t=0 \rightarrow t=2 \end{cases}$



$x(2t+1)$: $\begin{cases} x(0) \text{ @ } 2t+1=0 \rightarrow t=-\frac{1}{2} \\ x(1) \text{ @ } 2t+1=1 \rightarrow t=0 \end{cases}$

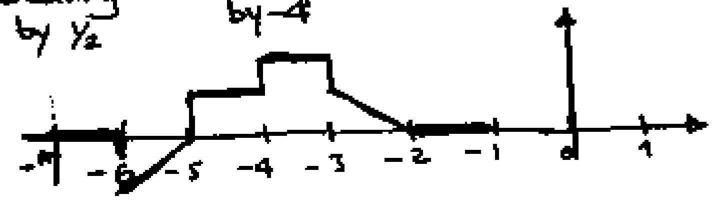
OR $x(2t+1) = S_2\, T_{-1}[x]$:

scale by 2 (S_2)

delay by -1 (prediction) (T_{-1})

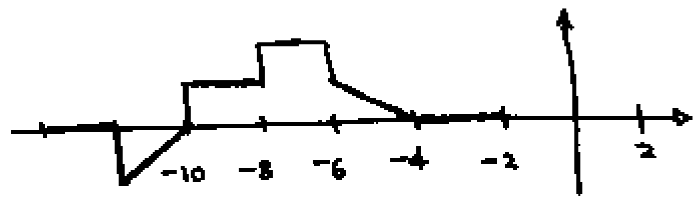
$$x(4 - t/2) = \underbrace{R}_{\text{Reflection}} \; \underbrace{S_{1/2}}_{\text{Scaling by } 1/2} \; \underbrace{T_{-4}}_{\text{Delay by } -4} [x]$$

$x \rightarrow \boxed{T_{-4}} \rightarrow$



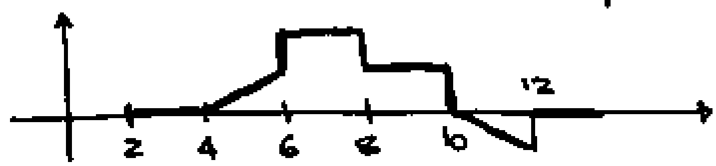
$\rightarrow \cdots$

$\cdots \rightarrow \boxed{S_{1/2}} \rightarrow$

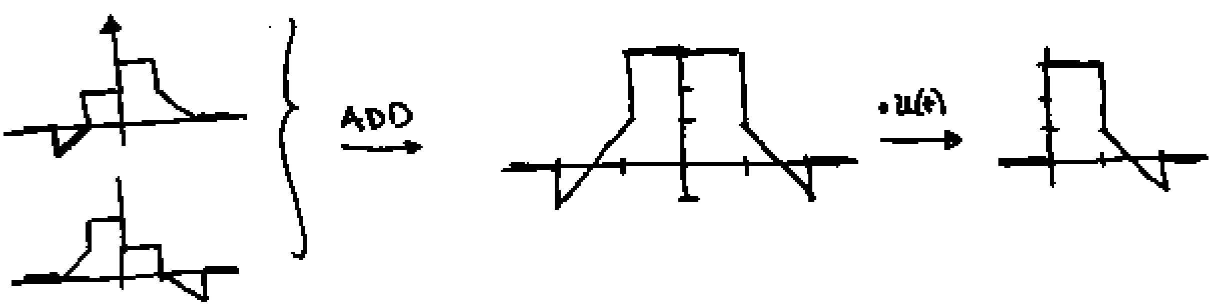


$\rightarrow \cdots$

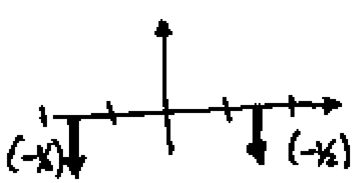
$\cdots \rightarrow \boxed{R} \rightarrow$



$\left[x(t) + x(-t)\right] u(t)$: sketch individually, add, multiply by $u(t)$



$$x(t)\left[\delta(t+3/2) - \delta(t-3/2)\right] = \underbrace{x(-3/2)}_{-1/2}\,\delta(t+3/2) - \underbrace{x(3/2)}_{1/2}\,\delta(t-3/2)$$



δ-impulses, weighted by $(-1/2)$

1.27

a) $y(t) = x(t-2) + x(2-t)$

Linearity : $\left[\alpha x_1(t-2) + \beta x_2(t-2)\right] + \left[\alpha x_1(2-t) + \beta x_2(2-t)\right]$

$= \left[\alpha x_1(t-2) + \alpha x_1(2-t)\right] + \left[\beta x_2(t-2) + \beta x_2(2-t)\right]$

$= \alpha y_1(t) + \beta y_2(t) \Rightarrow$ $\boxed{\text{System is Linear}}$

Time Invariance : We expect the system to be time-varying because of the reflection in $x(2-t)$. Formally, we need an example to verify that $HT_{t_0} \neq T_{t_0}H$ (for some input, some t_0 and at some t)

Other than trying some functions at random, we can examine $\Delta = HT_{t_0}[x] - T_{t_0}H[x]$ to locate the source of potential differences :

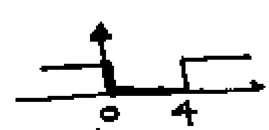
$HT_{t_0}[x] = x(t-t_0-2) + x(2-t-t_0)$

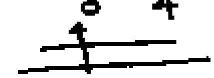
$T_{t_0}H[x] = x(t-t_0-2) + x(2-t+t_0)$

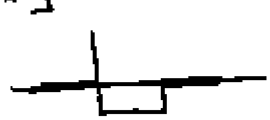
$\Rightarrow \Delta = x(2-t-t_0) - x(2-t+t_0)$

$\therefore$ we need a signal whose left and right shifts produce different values, i.e. not symmetric about 0.

TRY $x(t) = u(t)$

$\Rightarrow HT_{t_0}[x] =$ 

$T_{t_0}H[x] =$ 

$\Delta =$  $\neq 0 \Rightarrow$ $\boxed{\text{SYSTEM IS TIME VARYING}}$

Memory : $y(t)$ does not depend on $x(t)$ alone

eg $y(1)$ cannot be determined by knowing only $x(1)$

$\Rightarrow$ $\boxed{\text{system has memory}}$

CAUSALITY : we suspect non-causality because of the reflection.

Try $y(0) = x(-2) + x(2)$

future value

$\Rightarrow$ $\boxed{\text{SYSTEM IS NOT CAUSAL}}$

STABILITY : Suppose $|x(t)| \leq B \quad \forall t$

Then $|y(t)| \leq |x(t-2)| + |x(2-t)|$

$\leq B + B = 2B$ (bounded)

$\Rightarrow$ $\boxed{\text{SYSTEM is BIBO STABLE}}$

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b) $y(t) = \cos(3t)\, x(t)$

Linearity : $\cos(3t)[\alpha x_1(t) + \beta x_2(t)] = \alpha[\cos(3t)\, x_1(t)] + \beta[\cos(3t)\, x_2(t)]$

$= \alpha y_1(t) + \beta y_2(t) \Rightarrow$ $\boxed{\text{LINEAR}}$

Time Invariance : See Ex. 7 of class Notes $\rightarrow$ $\boxed{\text{TIME VARYING}}$

Memory : $y(t)$ depends only on $x(t)$ for all t $\rightarrow$ $\boxed{\text{MEMORYLESS}}$

Causality : Memoryless $\rightarrow$ $\boxed{\text{CAUSAL}}$

Stability: Let $|x(t)| < B$. Then $|y(t)| \leq |\cos 3t|\,|x(t)|$

$$\leq 1 \cdot |x(t)|$$
$$\leq B \Rightarrow \boxed{\text{BIBO STABLE}}$$

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c) $y(t) = \int_{-\infty}^{2t} x(\tau)\,d\tau$

Linearity: $\int_{-\infty}^{2t} [\alpha x_1(\tau) + \beta x_2(\tau)]\,d\tau = \alpha \int_{-\infty}^{2t} x_1(\tau)\,d\tau + \beta \int_{-\infty}^{2t} x_2(\tau)\,d\tau$

$$\Rightarrow \boxed{\text{LINEAR}}$$

Time Invariance: $\Delta = [HT_{t_0} - T_{t_0}H][x] = \int_{2t-2t_0}^{2t-t_0} x(\tau)\,d\tau$

Take $x(t) = u(t)$, $t_0 = -1$, $t = 0$

$$\Delta = \int_2^1 u(\tau)\,d\tau = -1 \neq 0 \Rightarrow \boxed{\text{TIME-VARYING}}$$

(Notice that this system is $\rightarrow \boxed{\int} \rightarrow \boxed{S_2} \rightarrow$)

TV element

Causality: $y(1)$ requires knowledge of $x[1,2]$ (future values)

$\Rightarrow$ SYSTEM IS $\boxed{\text{NOT CAUSAL}}$

Memory: Not causal $\Rightarrow \boxed{\text{Has Memory}}$

Stability: $x(t) = u(t)$ is a bounded input. But $y(t) = 2r(t)$ which is unbounded. $\Rightarrow \boxed{\text{UNSTABLE}}$

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d) $y(t) = \begin{cases} 0 & \text{if } t<0 \\ x(t)+x(t-2) & \text{if } t \geq 0 \end{cases}$

<u>Linearity</u>: As usual $\rightarrow$ $\boxed{\text{LINEAR}}$

<u>Time Invariance</u>: Rewrite $y(t) = [x(t)+x(t-2)]\,u(t)$ and work as in (b) $\Rightarrow$ $\boxed{\text{TIME VARYING}}$

time-varying multiplier

<u>Memory</u>: $y(t)$ depends on $x(t-2) \Rightarrow$ $\boxed{\text{HAS MEMORY}}$

<u>Causality</u>: $y(t)$ requires only present and past inputs $\Rightarrow$ $\boxed{\text{CAUSAL}}$

<u>Stability</u>: $|y(t)| \leq |x(t)| + |x(t-2)| \leq 2B \Rightarrow$ $\boxed{\text{BIBO STABLE}}$

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e) $y(t) = \begin{cases} 0 & \text{if } x(t)<0 \\ x(t)-x(t-2) & \text{if } x(t) \geq 0 \end{cases}$

<u>Linearity</u>: Switching depends on the input value.

Try $x(t) = u(t)$, $\alpha = -1$, $t = 1$

$$\left.\begin{aligned} H[\alpha x](1) &= \underset{\alpha x(1)=-1}{0} \\ \alpha H[x](1) &= \underset{x(1)=1}{(-1)}\,[x(1) - \cancel{x(-1)}^{\,0}] = -1 \end{aligned}\right\} \text{not equal}$$

$\Rightarrow$ $\boxed{\text{NONLINEAR}}$

<u>Time Invariance</u>: $T_{t_0} H[x] = y(t-t_0) = \begin{cases} 0 & \text{if } x(t-t_0)<0 \\ x(t-t_0)-x(t-t_0-2) & \text{otherwise} \end{cases}$

$$HT_{t_0}[x] = \begin{cases} 0 & \text{if } x(t-t_0)<0 \\ x(t-t_0)-x(t-t_0-2) & \text{otherwise} \end{cases}$$

$\Rightarrow HT_{t_0} = T_{t_0}H \Rightarrow$ $\boxed{\text{TIME INVARIANT}}$

<u>Memory</u> $y(t)$ depends on $x(t-2) \Rightarrow$ $\boxed{\text{Has MEMORY}}$
(at least if $x(1) > 0$)

<u>Causality</u> $y(t)$ requires only present or past values of $x \rightarrow$ $\boxed{\text{CAUSAL}}$

<u>Stability</u> $|y(t)| \leq \max\{0, |x(t)| + |x(t-2)|\} \leq 2B$
$\Rightarrow$ $\boxed{\text{BIBO STABLE}}$

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f) $y(t) = x(t/3)$

The system is $\boxed{\text{LINEAR}}$ (as usual)

It is $\boxed{\text{Time Varying}}$ (general result for scaling operations)

It is $\boxed{\text{not Causal}}$ (also a general result for scaling; notice $y(-1) = x(-1/3)$ a future x)

It $\boxed{\text{HAS MEMORY}}$ (since it is not causal)

It is $\boxed{\text{BIBO STABLE}}$ ($|y(t)| \leq B$)

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g) $y(t) = \frac{dx}{dt}(t) \triangleq \lim_{h\to 0} \frac{x(t+h) - x(h)}{h}$

<u>Linearity</u>: $\lim_{h\to 0} \frac{[\alpha x_1(t+h) + \beta x_2(t+h)] - [\alpha x_1(t) + \beta x_2(t)]}{h} =$

$= \alpha \lim_{h\to 0} \frac{x_1(t+h) - x_1(t)}{h} + \beta \lim_{h\to 0} \frac{x_2(t+h) - x_2(t)}{h}$

$\Rightarrow$ $\boxed{\text{LINEAR}}$

<u>Time Invariance</u>: $T_{t_0} H[x] = \lim_{h\to 0} \frac{x(t+h-t_0) - x(t-t_0)}{h}$

$H T_{t_0}[x] = \lim_{h\to 0} \frac{x(t-t_0+h) - x(t-t_0)}{h}$

$\Rightarrow T_{t_0} H = H T_{t_0} \Longrightarrow$ $\boxed{\text{TIME INVARIANT}}$

<u>Memory</u> The value of x at, say $t=1$, does not specify its slope at the same time instant $\rightarrow$ $\boxed{\text{HAS MEMORY}}$

<u>Causality</u> For continuously differentiable x, the left limit is the same as the right limit, so the slope can be determined with past information only.

<u>Stability</u> Let $x(t) = \sin t^2$. Then $y(t) = 2t \cos t^2$.
$y(t)$ is not bounded while $x(t)$ is bounded by 1.
$\Longrightarrow$ $\boxed{\text{UNSTABLE}}$

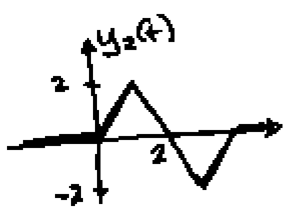
(<u>Note</u>: "$y(t)$ unbounded" does not require that $y(t)$ diverges monotonically.)

<u>#1.31</u> Observe that $x_2(t) = x_1(t) + (-1)\, x_1(t-2)$

Then $y_2(t) = H\left[x_1(t) + (-1)\, x_1(t-2) \right]$

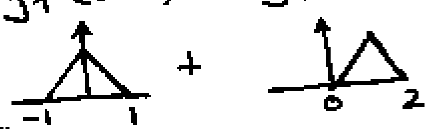
$= H[x_1(t)] + (-1)\, H[x_1(t-2)]$ (Linearity)

$= y_1(t) - y_1(t-2)$ (Time Invariance)



For $x_3(t) = x_1(t+1) + x_1(t)$

$\Rightarrow y_3(t) = y_1(t+1) + y_1(t)$

$=$ 

$\Rightarrow$ 

1.42 $\rightarrow \boxed{H_1} \rightarrow \boxed{H_2} \rightarrow$ $\qquad H_1, H_2$ are LTI

$H = H_2 H_1$

1. H_1 Linear $\Rightarrow H_1[\alpha x_1 + \beta x_2] = \alpha H_1[x_1] + \beta H_1[x_2]$

   H_2 Linear $\Rightarrow H_2[\alpha x_1 + \beta x_2] = \alpha H_2[x_1] + \beta H_2[x_2]$

$$\Rightarrow H_2[\underbrace{\alpha H_1[x_1] + \beta H_1[x_2]}_{H_1[\alpha x_1 + \beta x_2]}] = \alpha H_2[H_1[x_1]] + \beta H_2[H_1[x_2]]$$

$\Rightarrow$ <u>H is Linear.</u>

2. $$\left.\begin{array}{l} H_1 \text{ TI} \Rightarrow H_1 T_{t_0} = T_{t_0} H_1 \\ H_2 \text{ TI} \Rightarrow H_2 T_{t_0} = T_{t_0} H_2 \end{array}\right\} \Rightarrow H T_{t_0} = H_2 H_1 T_{t_0} = H_2 T_{t_0} H_1 = = T_{t_0} H_2 H_1 = T_{t_0} H$$

   $\Rightarrow$ <u>H is TI</u>

3. H_1, H_2 nonlinear. Then $H_2 H_1$ can be either Linear or nonlinear.

<u>Ex</u>: • $H_1: y(t) = x^3(t)$, $H_2: y(t) = x^{1/3}(t)$

$\Rightarrow H = H_2 H_1 \Rightarrow y(t) = H[x](t) = H_2[H_1[x]](t) = [x^3(t)]^{1/3} = x(t)$

($H_2 H_1$: composition)

$\Rightarrow$ H Linear.

• $H_1: y(t) = \sin x(t)$, $H_2: y(t) = x^2(t)$

$\Rightarrow H: y(t) = \sin^2 x(t)$ Nonlinear.

(Also, the "more likely" case)

$x \rightarrow \boxed{H_1} \xrightarrow{v} \boxed{H_2} \xrightarrow{w} \boxed{H_3} \xrightarrow{y}$

$H_1: \quad y(n) = \begin{cases} x(n/2) & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

$H_2: \quad y(n) = x(n) + \frac{1}{2} x(n-1) + \frac{1}{4} x(n-2)$

$H_3: \; y(n) = x(2n)$

$\Rightarrow$ Overall system : $y = H_3 H_2 H_1 [x]$
$= H_3 H_2 [v]$
$= H_3 [w]$

$\rightsquigarrow \; y(n) = H_3[w](n) = w(2n)$

$w(2n) = H_2[v](2n) = v(2n) + \frac{1}{2} v(2n-1) + \frac{1}{4} v(2n-2)$

$v(2n) = H_1[x](2n) = x(n)$

$v(2n-1) = H_1[x](2n-1) = 0$

$v(2n-2) = H_1[x](2n-2) = x(n-1)$

$\Rightarrow \; w(2n) = x(n) + \frac{1}{4} x(n-1)$

$$\boxed{y(n) = x(n) + \frac{1}{4} x(n-1) = H[x](n)}$$

$\Rightarrow$ H is Linear. [Verify that superposition holds ; also, H_1, H_2, H_3 are all linear and, by the first part of this problem, cascade connections of Linear systems are Linear

H is TI [$HT_{t_0}[x](n) = x(n-t_0) + \frac{1}{4} x(n-1-t_0)$

$T_{t_0} H[x](n) = x(n-t_0) + \frac{1}{4} x(n-t_0-1)$
$= HT_{t_0}[x](n)$

However, Notice that the shift does not commute with either H_1 or H_3. In this case it just happens that the time-variations of H_1 and H_3 "cancel".

Ch. 2 Selected HW Problems

2.12

$$y(t) = \{e^{-t}u(t)\} * \left\{ \sum_{k=-\infty}^{\infty} \delta(t-3k) \right\}$$

$$= \sum_k \{e^{-t}u(t)\} * \{\delta(t-3k)\}$$

$$= \cdots + e^{-(t+6)}u(t+6) + e^{-(t+3)}u(t+3) + e^{-t}u(t) + e^{-(t-3)}u(t-3) + \cdots$$

(since $t \in [0,3)$) $= \cdots + e^{-(t+6)}\underbrace{u(t+6)}_{1} + e^{-(t+3)}\underbrace{u(t+3)}_{1} + e^{-t}\underbrace{u(t)}_{1}$

$$= e^{-t}\underbrace{\left(1 + e^{-3} + e^{-6} + \cdots\right)}_{\text{Geometric series } \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}}$$

$$= e^{-t}\frac{1}{1-e^{-3}} . \quad \text{So, } \boxed{A = \frac{1}{1-e^{-3}}}$$

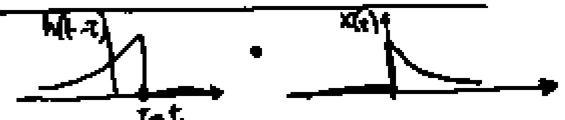
2.22 a. $x(t) = e^{-at}u(t)$, $h(t) = e^{-bt}u(t)$

$$y(t) = (h * x)(t)$$

$$= \int_{-\infty}^{\infty} e^{-b(t-\tau)}u(t-\tau)\, e^{-a\tau}u(\tau)\, d\tau$$

$$= \left[\int_0^t e^{-bt} e^{(b-a)\tau} d\tau\right] u(t)$$

Note Integrant:



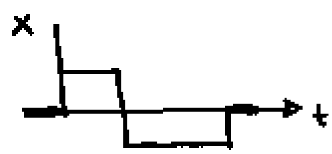
If $t<0$ the product is zero, hence the multiplication with $u(t)$. Without it, $\int_0^t e^{(b-a)\tau}d\tau = -\int_t^0 e^{(b-a)\tau}d\tau \neq 0$ producing an erroneus result.

Thus, $y(t) = \left[\int_0^t e^{(b-a)\tau} d\tau\right] e^{-bt} u(t)$

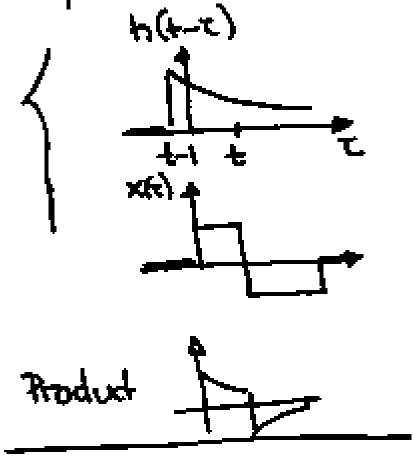
$$= \begin{cases} \dfrac{e^{(b-a)t} - 1}{b-a} & \text{if } b \neq a \\ t & \text{if } b = a \end{cases} \; e^{-bt} u(t)$$

$$= \begin{cases} \dfrac{e^{-at} - e^{-bt}}{b-a} u(t) & \text{if } b \neq a \\ t e^{-bt} u(t) & \text{if } b = a \end{cases}$$

b. $x(t) = u(t) - 2u(t-2) + u(t-5)$, $h(t) = e^{2t} u(1-t)$



$$y(t) = \int_{-\infty}^{\infty} e^{2(t-\tau)} u(1-t+\tau)\, x(\tau)\, d\tau$$



To simplify the computation, we may compute the response of the individual components of the input and add the results.

(That is, we use linearity to write
$x(t) = \sum_k x_k(t) \implies y(t) = \sum_k y_k(t)$ where $y_k = h * x_k$)

In this case, this is even easier since the system is TI and x_k are just shifted versions of each other. Effectively, we only need to compute $y_0 = h * u$.

CAVEAT The individual $h * x_k$ should exist for all k otherwise the decomposition is meaningless (e.g. produces $\infty - \infty = ?$)

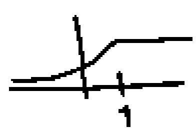
Let us define $y_0 = h * x$. Then,

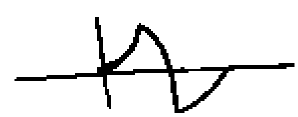
$$y_0(t) = \int_{-\infty}^{\infty} e^{2(t-\tau)} u(\tau - t + 1)\, x(\tau)\, d\tau$$

$$= \int_{\max(0,t-1)}^{\infty} e^{2(t-\tau)}\, d\tau = \frac{e^{2t} e^{-2\max(0,t-1)}}{2}$$

Then, by Linearity and Time Invariance,

$$y(t) = y_0(t) + y_0(t-5) - 2\, y_0(t-2)$$

$y_0(t) =$ 

, $y(t) =$ 

In MATLAB:

```
t = [-10 : 0.1 : 10]';
y0 = min( exp(2*t), exp(2)) /2;
y1 = min( exp(2*(t-2)), exp(2)) /2;
y2 = min( exp(2*(t-5)), exp(2)) /2;
plot (t, y0-2*y1 + y2)
```

C. $h(t) = u(t-1) - u(t-3)$, $x(t) = u(t)\sin \pi t - u(t-2)\underbrace{\sin(\pi t)}_{=\sin(\pi(t-2))}$

Again, use the LTI ppties to write

$$y(t) = y_0(t) - y_0(t-2)$$

where $y_0(t) = \int_{-\infty}^{\infty} h(t-\tau)\, u(\tau)\sin(\pi\tau)\, d\tau$

$$= \int_0^{\infty} h(t-\tau) \sin \pi\tau\, d\tau.$$

We can now integrate the individual components of h

$$y_0(t) = \int_0^\infty u(t-\tau-1)\sin\pi\tau\,d\tau - \int_0^\infty u(t-\tau-3)\sin\pi\tau\,d\tau$$

$$= \left(\int_0^{t-1}\sin\pi\tau\,d\tau\right)u(t-1) - \left(\int_0^{t-3}\sin\pi\tau\,d\tau\right)u(t-3)$$

don't forget

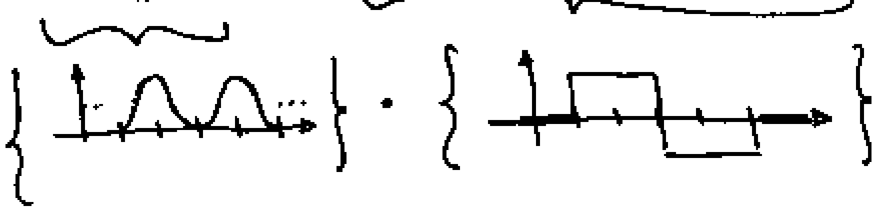
$$= \frac{1-\cos(\pi t-\pi)}{\pi}\,u(t-1) - \frac{1-\cos(\pi t-3\pi)}{\pi}\,u(t-3)$$

$$= \frac{1-\cos(\pi t-\pi)}{\pi}\left(u(t-1)-u(t-3)\right)$$

$$= \frac{1+\cos\pi t}{\pi}\left(u(t-1)-u(t-3)\right)$$

$$\Rightarrow\quad y(t) = y_0(t) - y_0(t-2) = \cdots =$$

$$= \frac{1+\cos\pi t}{\pi}\left(u(t-1)-2u(t-3)+u(t-5)\right)$$



$=$

d. $x(t) = at+b$, $\quad h(t) = \underbrace{-\frac{1}{3}\delta(t-2)}_{h_1} + \underbrace{\frac{4}{3}\left[u(t)-u(t-1)\right]}_{h_2}$

$$y = h * x = h_1 * x + h_2 * x.$$

$$y_1(t) = \int_{-\infty}^{\infty} -\frac{1}{3}\delta(t-\tau-2)(a\tau+b)\,d\tau$$

$$= -\frac{1}{3}(a\tau+b)\Big|_{\tau=t-2} \int_{-\infty}^{\infty} \delta(t-\tau-2)\,d\tau$$

$$= -\frac{1}{3}\left[a(t-2)+b\right] \longrightarrow \text{shifted } at+b.$$

$$y_2(t) = \frac{4}{3}\int_{-\infty}^{\infty}\left[u(t-\tau) - u(t-\tau-1)\right](a\tau+b)\,d\tau$$

Note Recall the CAVEAT! This is a case where we cannot decompose h_2 any further:

$$\int_{-\infty}^{\infty} u(t-\tau)(a\tau+b)\,d\tau = \int_{-\infty}^{t}(a\tau+b)\,d\tau \rightarrow \text{diverges!}$$

$$y_2(t) = \frac{4}{3}\int_{t-1}^{t}(a\tau+b)\,d\tau = \frac{4}{3}\left[\frac{a}{2}\tau^2\Big|_{t-1}^{t} + b\tau\Big|_{t-1}^{t}\right]$$

$$= \frac{4}{3}\left[\frac{a}{2}(2t-1) + b\right]$$

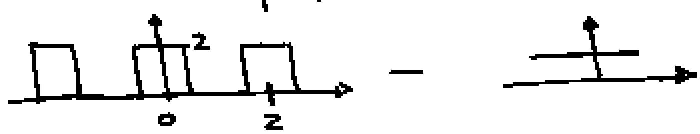
Finally, $y(t) = y_1(t) + y_2(t) = -\frac{1}{3}\left[a(t-2)+b\right] + \frac{4}{3}\left[\frac{a}{2}(2t-1)+b\right]$

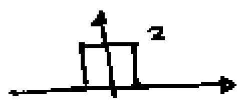
$$= \underline{at+b}$$

Note We found that $\xrightarrow{at+b}\boxed{H}\xrightarrow{at+b}$. This does not imply that $\xrightarrow{x(t)}\boxed{H}\xrightarrow{x(t)}$ for all x. That is, it is not necessarily $H = I$.

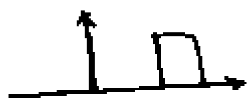
e. $h(t) = r(-t+1)\,u(t)$, $r(t) = t\,u(t)$

$x(t) = \dots$ many possible expressions. Personal preference





$= 2\left[u(t+0.5) - u(t-0.5)\right] = x_0(t)$



$= x_0(t-2)$, etc.

$\Rightarrow x(t) = \left[\sum_{k=-\infty}^{\infty} x_0(t-2k)\right] - 1.$

Define $y_0 = h * x_0$, $y_1 = h * x_1$ $\left(x_1(t) = 1\right)$

Then (LTI) $y(t) = \sum_{k=-\infty}^{\infty} y_0(t-2k) - y_1(t)$

Compute $y_1(t) = \int_{-\infty}^{\infty} x_1(t-\tau)\,h(\tau)\,d\tau$

$= \int_{-\infty}^{\infty} h(\tau)\,d\tau = \frac{1}{2}$ (by inspection; sketch $h(t)$ vs t)

Compute $y_0(t) = \int_{-\infty}^{\infty} x_0(t-\tau)\,h(\tau)\,d\tau$

$= \int_{t-0.5}^{t+0.5} 2\,u(\tau)(1-\tau)\,u(1-\tau)\,d\tau$

Distuguish cases: $t+0.5 < 0$ $u(\tau) = 0 \Rightarrow \int = 0$

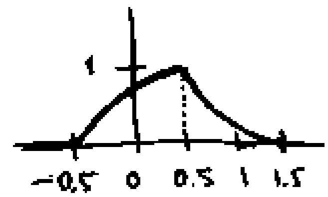
$t-0.5 > 1$ $u(1-\tau) = 0 \Rightarrow \int = 0$

$1 > t+0.5 > 0 \Rightarrow \int_0^{t+0.5} \dots$

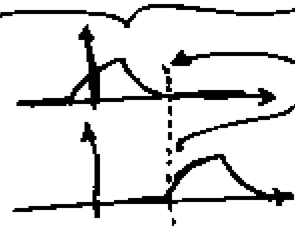
$1 > t-0.5 > 0 \Rightarrow \int_{t-0.5}^{1} \dots$

$$\Rightarrow \quad y_0(t) = \begin{cases} 0 & \text{if } t < -0.5 \\ 2\int_0^{t+0.5} (1-\tau)\,d\tau & \text{if } -0.5 < t < 0.5 \\ 2\int_{t-0.5}^{1} (1-\tau)\,d\tau & \text{if } 0.5 < t < 1.5 \\ 0 & \text{if } t > 1.5 \end{cases}$$

$$= \begin{cases} 0 & \cdots \\ (t+\tfrac{1}{2})(\tfrac{3}{2}-t) & \cdots \\ (\tfrac{3}{2}-t)(\tfrac{3}{2}-t) & \cdots \\ 0 & \cdots \end{cases} =$$

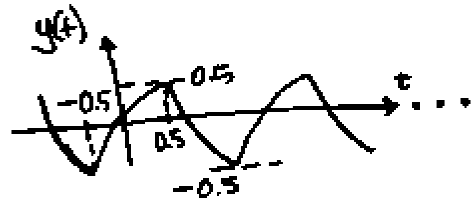


Then $$y(t) = \sum_{k=-\infty}^{\infty} y_0(t-2k) - \tfrac{1}{2}$$



Fortunately, no overlap

etc

$=$ 

2.29

a. Causal because $h(t) = 0$ for $t < 0$.
Stable because $\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{2}^{\infty} e^{-4t}\,dt = \frac{e^{-8}}{4} < \infty$

b. $h(t) = e^{-6t}u(3-t)$: Not causal because $h(t) \neq 0$ for some $t < 0$.
unstable because $\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{-\infty}^{3} e^{-6t}\,dt$ <u>diverges</u>

c. Not causal $\left(u(t+50) \right)$
Stable $\left(\int |h| = \int_{-50}^{\infty} e^{-2t}\,dt = \frac{e^{100}}{2} < \infty \right)$

d. Not causal $\left(u(-t-1) \right)$
Stable $\left(\int |h| = \int_{-\infty}^{-1} e^{2t}\,dt = \frac{1}{2e^{2}} < \infty \right)$

e. Not causal $\left(e^{-6|t|} \neq 0 \text{ for } t < 0 \right)$
Stable $\left(\int |h| = \int_{-\infty}^{0} e^{6t}\,dt + \int_{0}^{\infty} e^{-6t}\,dt = \frac{1}{6} + \frac{1}{6} < \infty \right)$

f. Causal $\left(u(t) \right)$
Stable ($\int |h| = \ldots = 1 < \infty$; or h = polynomial × decaying exponential → absolutely integrable)

g. Causal, Stable

2.40

a. $y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau-2)\, d\tau.$

Impulse response $h(t) = H[\delta(t)]$

$$= \int_{-\infty}^{t} e^{-(t-\tau)} \delta(\tau-2)\, d\tau$$

$$= \begin{cases} e^{-(t-2)} & \text{if } t>2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \boxed{e^{-(t-2)}\, u(t-2)}$$

Alternative way: rewrite $y(t)$ in the form $\int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$

$$y(t) = \int_{-\infty}^{\infty} \{e^{-(t-\tau)} x(\tau-2)\} u(t-\tau)\, d\tau$$

$$\overset{\tau'=\tau-2}{=} \int_{-\infty}^{\infty} u(t-\tau'-2)\, e^{-(t-\tau'-2)} x(\tau')\, d\tau'.$$

$\Rightarrow h(t-\tau) = u(t-\tau-2)\, e^{-(t-\tau-2)}$

$\Rightarrow \boxed{h(t) = u(t-2)\, e^{-(t-2)}}$

b. $x(t) = u(t+1) - u(t-2)$

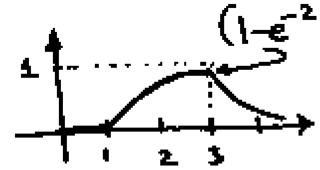
Compute step response first, then use LTI ppties.

$$y_0(t) = \int_{-\infty}^{\infty} h(t-\tau)\, u(\tau)\, d\tau = \int_{-\infty}^{\infty} e^{-(t-\tau-2)} u(t-\tau-2)\, u(\tau)\, d\tau$$

$$= \left[\int_{0}^{t-2} e^{-(t-\tau-2)}\, d\tau\right] u(t-2) = \cdots = \left[1-e^{-(t-2)}\right]u(t-2)$$

LTI $\Rightarrow$ $y(t) = y_0(t+1) - y_0(t-2)$

$$= \left[1-e^{-(t-1)}\right]u(t-1) - \left[1-e^{-(t-3)}\right]u(t-3)$$



2.45

a. $y = H[x]$, $y_1 = H\left[\frac{dx}{dt}\right]$

$$= H\left[\lim_{\Delta\to 0} \frac{x(t) - x(t-\Delta)}{\Delta}\right]$$

$$= H\left[\lim_{\Delta\to 0} \frac{x(t) - T_\Delta x(t)}{\Delta}\right] \quad \text{(introduce shift}$$

$$= \lim_{\Delta\to 0} \frac{1}{\Delta}\left(H[x] - HT_\Delta[x]\right) \quad \text{(use Linearity}$$

$$= \lim_{\Delta\to 0} \frac{1}{\Delta}\left(H[x] - T_\Delta H[x]\right) \quad \text{(Use TI}$$

$$= \lim_{\Delta\to 0} \frac{1}{\Delta}\left(y(t) - y(t-\Delta)\right)$$

$$= \frac{dy}{dt}$$

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau.$$

$$\frac{dy}{dt}(t) = \frac{d}{dt}\int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\frac{d}{dt}x(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\left[\frac{dx}{dt}\right](t-\tau)\,d\tau \qquad = \frac{d}{d(t-\tau)}x(t-\tau)$$

$\frac{d}{dt}$: LTI system $\rightsquigarrow$ described by a convolution with impulse response $u_1(t) = \frac{d\delta}{dt}$ (Unit doublet)

$$x(t) = \int_{-\infty}^{\infty} \delta(t-\tau)\,x(\tau)\,d\tau$$

$$\frac{dx}{dt}(t) = \int_{-\infty}^{\infty} \frac{d\delta}{dt}(t-\tau)\,x(\tau)\,d\tau = \int_{-\infty}^{\infty} u_1(t-\tau)\,x(\tau)\,d\tau$$

$$y_1(t) = (h * \dot{x})(t) = \left(h * (u_1 * x)\right)(t) = \left((h * u_1) * x\right)(t)$$

$$= \left(u_1 * (h * x)\right)(t) = (u_1 * y)(t)$$

$$= \frac{dy}{dt}(t)$$

---

b. from the last part, $y_1 = (u_1 * h) * x = \frac{dy}{dt}$

But $u_1 * h = \frac{dh}{dt} \Rightarrow \underline{\mathring{y} = \mathring{h} * x}$.

Next, $y = h * x$, $\int_{-\infty}^{t} x(\tau)\,d\tau = (u * x)(t)$

$$\Rightarrow u * x * \mathring{h} = (u * \mathring{h}) * x = h * x = y$$

Note: $u * \mathring{h} = u * (u_1 * h) = (u * u_1) * h = \delta * h$
$= h$

$$\int_{-\infty}^{t} (\mathring{x} * h)(\tau)\,d\tau = \big(u * (\mathring{x} * h)\big)(t)$$
$$= \big((u * u_1) * (x * h)\big)(t)$$
$$= \big(\delta * (x * h)\big)(t) = (x * h)(t)$$
$$= y(t)$$

$$\mathring{x} * \left(\int_{-\infty}^{t} h(\tau)\,d\tau\right) = \mathring{x} * (u * h) = (x * u_1) * (u * h)$$
$$= x * (u_1 * u) * h = x * \delta * h = x * h$$
$$= y.$$

---

c. $\sin\omega_0 t = \int_{-\infty}^{\infty} h(t-\tau)\, e^{-5\tau} u(\tau)\,d\tau$

$$\frac{d}{dt}\sin\omega_0 t = \omega_0 \cos\omega_0 t = \int_{-\infty}^{\infty} h(t-\tau)\frac{d}{d\tau}\left[e^{-5\tau}u(\tau)\right]d\tau$$
$$= \underbrace{\int_{-\infty}^{\infty} h(t-\tau)\left[-5e^{-5\tau}u(\tau)\right]d\tau}_{-5\sin\omega_0 t} + \underbrace{\int_{-\infty}^{\infty} h(t-\tau)e^{-5\tau}\delta(\tau)\,d\tau}_{h(t)}$$

$\Rightarrow \omega_0\cos\omega_0 t = -5\sin\omega_0 t + h(t)$

$\Rightarrow \boxed{h(t) = \omega_0\cos\omega_0 t + 5\sin\omega_0 t}$

d. $y = h*x, \quad s = h*u$

But $u_1 * u = \delta$ and $y = \delta * y$

$$\therefore \quad y = h*x*\delta$$
$$= h*x*u_1*u$$
$$= (h*u)*(u_1*x)$$
$$= s*\dot{x}$$

Similarly, $x = \delta * x = u*u_1*x = u*\dot{x}$

$\left(\dot{x} = \frac{dx}{dt}\right)$

---

e. $s(t) = \left(e^{-3t} - 2e^{-2t} + 1\right)u(t), \quad x(t) = e^{t}u(t)$

$$y = h*x = s*\dot{x}$$

$$\dot{x}(t) = e^{t}u(t) + e^{t}\delta(t)$$
$$= e^{t}u(t) + \delta(t)$$

$$\Rightarrow y(t) = (s*\delta)(t) + (s*[e^{t}u(t)])(t)$$
$$= s(t) + \int_{-\infty}^{\infty} e^{(t-\tau)}u(t-\tau)\left(e^{-3\tau} - 2e^{-2\tau} + 1\right)u(\tau)\,d\tau$$
$$= \left(e^{-3t} - 2e^{-2t} + 1\right)u(t) + u(t)e^{t}\int_0^t \left(e^{-\tau} + e^{-4\tau} - 2e^{-3\tau}\right)d\tau$$
$$= \left(e^{-3t} - 2e^{-2t} + 1\right)u(t) + u(t)e^{t}\left[1 - e^{-t} + \frac{1}{4}(1 - e^{-4t}) - \frac{2}{3}(1 - e^{-3t})\right]$$
$$= \left\{\frac{7}{12}e^{t} - \frac{4}{3}e^{-2t} + \frac{3}{4}e^{-3t}\right\}u(t)$$

#2.48

a. Impulse response h, periodic, nonzero.

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \lim_{T\to\infty} \int_{-T}^{T} |h(t)|\,dt$$

Let T_0 be the period of $h(t)$, i.e., $h(t+T_0) = h(t)$.

Then $\int_0^{KT_0} |h(t)|\,dt = K \int_0^{T_0} |h(t)|\,dt$

The last integral is a positive number since $h(t) \neq 0$,

say $\int_0^{T_0} |h(t)|\,dt = a > 0$.

Now, let K be the integer part of $\frac{T}{T_0}$, ie.

$T \geq KT_0$ and $T < (K+1)T_0$.

Then, $\int_{-T}^{T} |h(t)|\,dt \geq \int_{-KT_0}^{KT_0} |h(t)|\,dt = 2Ka$.

As $T \to \infty$, $K \to \infty$. This implies

$$\lim_{T\to\infty} \int_{-T}^{T} |h(t)|\,dt \geq \lim_{k\to\infty} 2Ka = \infty$$

So h is not absolutely integrable ($\int |h|$ diverges)

and the system is $\boxed{\text{not BIBO stable}}$ (a is true)

c. $|h(n)| \leq k \not\Rightarrow$ stability. (c is a false statement)

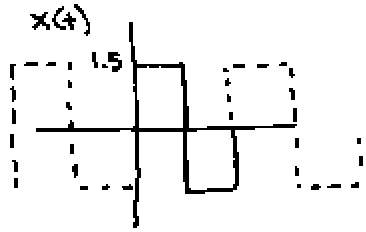
Example: $h(n) = u(n)$ bounded by 1.

But $u * u = r$, the ramp function which is unbounded. (Same argument for both Continuous and discrete time.)

d. The continuous time version requires more conditions e.g. $|h(t)| \leq K$. (This holds in discrete time for finite duration sequences.)

CH.3. SELECTED HW SOLUTIONS

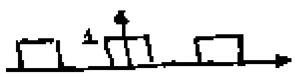
 # 3.4



$T = 2$

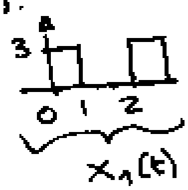
$\omega_0 = \frac{2\pi}{2} = \pi$

The Fourier Series coefficients for the periodic square wave

$x_0 =$ 

have been derived in eq. 3.39.

Here, we use this formula and some signal manipulations to compute $FS\{x(t)\}$.

$x(t) = -1.5 +$ 

, $x_1(t) = 3x_0(t - \tfrac{1}{2})$

Using FS properties,

$$FS\{x_1\} = 3\,FS\{x_0(t-\tfrac{1}{2})\} = 3e^{-jk\omega_0 \frac{1}{2}} FS\{x_0\}$$

$$FS\{x_0\} = \begin{cases} \frac{1}{2} & k=0 \\ \frac{\sin(k\omega_0 \frac{1}{2})}{k\pi} & k\neq 0 \end{cases} = \begin{cases} \frac{1}{2} & k=0 \\ \frac{\sin(k\pi/2)}{k\pi} & k\neq 0 \end{cases}$$

$$\Rightarrow a_k = FS\{x(t)\} = FS\{-1.5\} + FS\{x_1(t)\}$$

$$= \begin{cases} -1.5 & k=0 \\ 0 & k\neq 0 \end{cases} + 3e^{-jk\pi/2} \begin{cases} \frac{1}{2} & k=0 \\ \frac{\sin(k\pi/2)}{k\pi} & k\neq 0 \end{cases}$$

$$\Rightarrow \boxed{a_k = 3e^{-jk\pi/2} \frac{\sin k\pi/2}{k\pi} \quad k\neq 0}$$

Compute a_0 separately (GOOD PRACTICE) $a_0 = \frac{1}{T}\int_T x(t)\,dt = 0$

$$\boxed{a_0 = 0}$$

#3.8

$x(t)$ real and odd, period $T=2$

$a_k = 0$ for $|k|>1$

$\frac{1}{2}\int_0^2 |x(t)|^2 dt = 1$

$\Rightarrow$ Real + odd $\Rightarrow$ $a_0 = 0$, a_k pure imaginary and odd.

$a_k = 0$ for $|k|>1$ $\Rightarrow$ $x(t) = a_{-1}e^{-j\omega_0 t} + a_1 e^{j\omega_0 t}$

a_k odd $\Rightarrow$ $a_k = -a_{-k}$ $\Rightarrow$ $x(t) = a_1\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$

$\frac{1}{2}\int_0^2 |x|^2 = 1 \Rightarrow \sum |a_k|^2 = 2|a_1|^2 = 1 \Rightarrow |a_1| = \frac{1}{\sqrt{2}}$

a_1 imaginary $\Rightarrow$ $a_1 = j/\sqrt{2}$ or $a_1 = -j/\sqrt{2}$

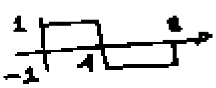
$\Rightarrow$ $\boxed{x(t) = \sqrt{2}\sin\omega_0 t}$ or $\boxed{x(t) = -\sqrt{2}\sin\omega_0 t}$ $(\omega_0 = \pi)$

#3.13

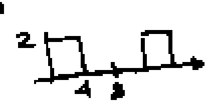
$x(t) = \sum a_k e^{jk\omega_0 t} \rightarrow \boxed{H} \rightarrow y(t) = \sum b_k e^{jk\omega_0 t}$

Key ppty for LTI systems : $\boxed{b_k = a_k H(jk\omega_0)}$

$H(j\omega)$: transfer function $= \mathcal{F}\{h(t)\}$.

$x(t) =$  $T=8$, $\omega_0 = \pi/4$

Working as in pr. 3.4, $a_0 = 0$

$x_1 =$ , $Fs\{x_1\} = \frac{\sin k\omega_0 2}{k\pi} \cdot 2e^{-jk\omega_0 2}$ for $k \neq 0$

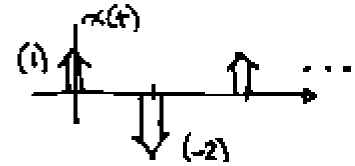
Thus, $a_k = \begin{cases} 0 & \text{if } k=0 \\ 2e^{-jk\pi/2}\frac{\sin k\pi/2}{k\pi} & \text{if } k\neq 0 \end{cases}$

$H(jk\omega_0) = H\left(j\frac{k\pi}{4}\right) = \frac{4\sin k\pi}{k\pi}$

$\Rightarrow$ $b_k = \begin{cases} 0 & \text{if } k=0 \\ 8e^{-jk\pi/2}\frac{\sin k\pi/2 \, \sin k\pi}{k^2\pi^2} & \text{if } k\neq 0 \end{cases}$, and $y(t) = \sum_{k=-\infty}^{\infty} b_k e^{j\frac{k\pi}{4}t}$

But in this case $\sin k\pi = 0 \;\forall k$ $\Rightarrow$ $b_k = 0$ $\Rightarrow$ $\boxed{y(t) = 0}$

3.22

d. 

$T = 2$, $a_0 = \frac{1}{2}\int_{-0.5}^{1.5} x(t)\,dt = \frac{1-2}{2} = -\frac{1}{2}$

↖ avoid δ at the boundary

For the rest of the coefficients, we write

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t-2n) - 2\sum_{n=-\infty}^{\infty} \delta(t-2n-1)$$

$$Fs\{x\} = Fs\{\sum \delta(t-2n)\} - 2e^{-jk\omega_0 \cdot 1}\, Fs\{\sum\delta(t-2n)\}$$

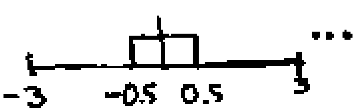
$$= \left[1 - 2e^{-jk\pi}\right]\cdot\frac{1}{2}$$

↖ $\frac{1}{T}$ = coeff. for the δ-train

$$= \left(\frac{1}{2} - e^{-jk\pi}\right) \text{ for } k \neq 0$$

We may also observe that $e^{-jk\pi} = \begin{cases} 1 & \text{if } k \text{ even} \\ -1 & \text{if } k \text{ odd}\end{cases} \Rightarrow \boxed{a_k = \begin{cases} -\frac{1}{2} & k \text{ even} \\ \frac{3}{2} & k \text{ odd}\end{cases}}$

---

e. Define $x_0(t) =$ 

$T = 6$
$\omega_0 = \pi/3$

Then, $x(t) = x_0(t+1.5) - x_0(t-1.5)$

But $Fs\{x_0\} = \begin{cases} 1/6 & k=0 \\ \frac{\sin k\pi/6}{k\pi} & k\neq 0\end{cases}$

$$\Rightarrow Fs\{x\} = \begin{cases} 0 & k=0 \\ e^{jk\pi/2}\frac{\sin k\pi/6}{k\pi} - e^{-jk\pi/2}\frac{\sin k\pi/6}{k\pi} \end{cases}$$

Further simplification (easy for this problem):

$$\frac{\sin k\pi/6}{k\pi}\left(e^{jk\pi/2} - e^{-jk\pi/2}\right) = \underbrace{2j}\,\frac{\sin(k\pi/6)\sin(k\pi/2)}{k\pi}$$

Notice: Real-odd signal $\Rightarrow$ a_k imaginary-odd functions of k

#3.5

Both $x_1(1-t)$ and $x_1(t-1)$ are periodic with fundamental period $T_1 = \frac{2\pi}{\omega_1}$. Since $y(t)$ is a linear combination of the two, it is also periodic with fundamental period $T_2 = \frac{2\pi}{\omega_1}$. Therefore, $\omega_2 = \omega_1$.

Now, $\mathcal{F}s\{x_1(t+1)\} = a_k e^{jk\frac{2\pi}{T_1}}$

$\mathcal{F}s\{x_1(t-1)\} = a_k e^{-jk\frac{2\pi}{T_1}} \Rightarrow \mathcal{F}s\{x_1(-t+1)\} = a_{-k} e^{-jk\frac{2\pi}{T_1}}$

Hence, $\mathcal{F}s\{x_1(t+1) + x_1(1-t)\} = a_k e^{jk\frac{2\pi}{T_1}} + a_{-k} e^{-jk\frac{2\pi}{T_1}}$

$$= \boxed{e^{-jk\omega_1}(a_k + a_{-k})}$$

#3.34

Compute $H(j\omega)$ first:

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt = \int_0^{\infty} e^{-4t} e^{-j\omega t} dt + \int_{-\infty}^{0} e^{4t} e^{-j\omega t} dt$$

$$= \frac{-1}{4+j\omega}[0-1] + \frac{-1}{4-j\omega}[0-1] = \frac{1}{4+j\omega} + \frac{1}{4-j\omega}$$

$$= \frac{4-j\omega+4+j\omega}{16+\omega^2} = \frac{8}{16+\omega^2}$$

Next, compute the Fourier coefficients for each sequence and apply the filtering property:

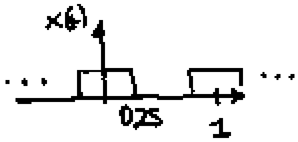
a. $x(t) = \sum_n \delta(t-n) = \ldots \quad T = 1,\ \omega_0 = 2\pi$

$a_k = \frac{1}{1}\int_{-1/2}^{1/2} \delta(t) e^{-jk\omega_0 t} dt = 1 \Rightarrow x(t) = \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$

$y(t) = (h * x)(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$; $b_k = H(jk\omega_0)\, a_k = 1 \cdot \frac{8}{16+4\pi^2 k^2}$

$$\Rightarrow y(t) = \sum_{k=-\infty}^{\infty} \frac{8}{16+4\pi^2k^2} e^{jk\omega_0 t} = \boxed{\sum_k \frac{2}{4+\pi^2k^2} e^{j2\pi kt}}$$

---

c. 

$T = 1, \quad \omega_0 = 2\pi$

From example 3.5 (or Tables) $a_k = \dfrac{\sin k\pi/2}{k\pi} \quad k \neq 0$

$a_0 = 1/2$

$y(t) = (h*x)(t) = \sum_k b_k e^{jk\omega_0 t}$; $b_k = H(jk\omega_0)\, a_k$

$$\Rightarrow \boxed{y(t) = \underbrace{\frac{1}{4}}_{b_0} + \sum_{k\neq 0} \frac{\sin k\pi/2}{k\pi} \frac{2}{4+k^2\pi^2} e^{j2\pi k t}}$$

3.46

a. $x(t) = \sum_k a_k e^{jk\omega_0 t}$; $y(t) = \sum_k b_k e^{jk\omega_0 t}$

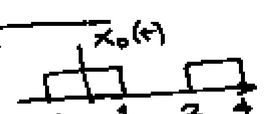
$$x(t)y(t) = \sum_n a_n e^{jn\omega_0 t} \sum_\ell b_\ell e^{j\ell\omega_0 t}$$

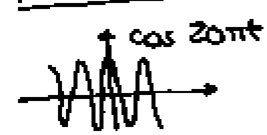
$$= \sum_n \sum_\ell a_n b_\ell e^{j(n+\ell)\omega_0 t}$$; regroup terms with the same exponential $n+\ell = k$

$$= \sum_n \sum_k a_n b_{k-n} e^{jk\omega_0 t}$$

$$= \sum_k \left[\sum_n a_n b_{k-n}\right] e^{jk\omega_0 t}$$

$$= \sum_k c_k e^{jk\omega_0 t}$$ where $\boxed{c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}}$

b. $x_1(t) = \Big($ 

$\Big) \cdot \Big($ 

$\Big)$; $T = 3$, $\omega_0 = \frac{2\pi}{3}$

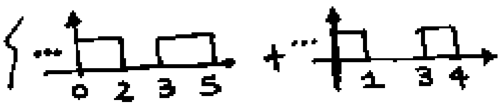
$a_k = Fs\{x_0\} \overset{Ex.3.5}{=} \ldots$

$b_k = Fs\{\cos 20\pi t\}$; $\cos 20\pi t = \dfrac{e^{j20\pi t} + e^{-j20\pi t}}{2} = \dfrac{e^{j30\frac{2\pi}{3}t} + e^{-j30\frac{2\pi}{3}t}}{2}$

$\Rightarrow b_k = \begin{cases} 1/2 & \text{if } |k| = 30 \\ 0 & \text{otherwise.} \end{cases}$

Now, apply a. $c_k = \sum_n a_n b_{k-n} = \sum_n a_{k-n} b_n = \boxed{\frac{1}{2}\left(a_{k+30} + a_{k-30}\right)}$

$\neq 0$ for $n = \pm 30$ only

Similarly for $x_2(t)$ where $a_k = FS\{$  $\}$

(Use Example 3.5 twice and apply the corresponding shifts...)

$x_3(t)$: $a_k = FS\{$  $\}$ $T = 4$, $\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

$$a_k = \frac{1}{T}\int_{-1}^{1} e^{-|t|} e^{-jk\omega_0 t}\,dt = \frac{1}{T}\left[\int_0^1 e^{-(1+jk\omega_0)t}dt + \int_{-1}^{0} e^{(1-jk\omega_0)t}dt\right]$$

$$= \frac{1}{4}\frac{-1}{1+jk\omega_0}\left[e^{-(1+jk\omega_0)}-1\right] + \frac{1}{4}\frac{1}{1-jk\omega_0}\left[1-e^{-(1-jk\omega_0)}\right]$$

etc.

---

c. If $y = x^*$ then $b_k = a_{-k}^*$ (from pptics Table)

$z(t) = |x(t)|^2 = x(t)\,y(t) = \sum_k c_k e^{jk\omega_0 t}$, where $c_k = \sum_n a_n a_{n-k}^*$

But $c_0 = \frac{1}{T}\int_0^T z(t)\,e^0\,dt$. Substitute to get

$$c_0 = \sum_n a_n a_n^* = \sum_n |a_n|^2 = \frac{1}{T}\int_0^T |x(t)|^2 dt$$

 # 3.63 $\quad x(t) = \sum_k a^{|k|} e^{jk\frac{\pi}{4}t}$; $a \in (0,1)$

$$H(j\omega) = \begin{cases} 1 & \text{if } |\omega| \le W \\ 0 & \text{otherwise.} \end{cases}$$

$\therefore\ y(t) = \sum_k a^{|k|} H\left(j\frac{k\pi}{4}\right) e^{-j\frac{k\pi}{4}t}$.

Let $K_0 = \left[\frac{4W}{\pi}\right]$ (the integer part). Then $y(t) = \sum_{k=-K_0}^{K_0} a^{|k|} e^{jk\frac{\pi}{4}t}$.

Now, $E(x) = \frac{1}{T}\int_T |x|^2 = \sum_k a^{2|k|} = \sum_{k=0}^{\infty} a^{2|k|} + \sum_{k=0}^{\infty} a^{2|-k|} - 1$

$$= \frac{2}{1-a^2} - 1 = \frac{1+a^2}{1-a^2}$$

Further, $E(y) = \frac{1}{T}\int_T |y|^2 = \sum_{k=-K_0}^{K_0} a^{2|k|} = 2\sum_{k=0}^{K_0} a^{2k} - 1 = \frac{2\left[1-a^{2(K_0+1)}\right]}{1-a^2} - 1$

We want $E(y) \geq 0.9\,E(x)$ or $E(x) - E(y) \leq 0.1\,E(x)$

Substituting, $\dfrac{1+a^2}{1-a^2} - \dfrac{1+a^2-2a^{2(K_0+1)}}{1-a^2} \leq 0.1\,\dfrac{1+a^2}{1-a^2}$

$$\Rightarrow \frac{2a^{2(K_0+1)}}{1-a^2} \leq 0.1\,\frac{1+a^2}{1-a^2}$$

since $|a|<1$ $\Rightarrow$ $a^{2(K_0+1)} \leq \dfrac{1+a^2}{20}$ $\Rightarrow (\ln a^2)(K_0+1) \leq \ln\left(\dfrac{1+a^2}{20}\right)$

both ln: -ve $\Rightarrow$ $\boxed{K_0 \geq \dfrac{\ln\left(\frac{1+a^2}{20}\right)}{\ln a^2} - 1}$

 # 3.47 $x(t) = \cos\omega_0 t$, Fundamental period $T_0 = \frac{2\pi}{\omega_0}$.

It is also periodic with period $T = nT_0$, $n = 1, 2, \ldots$

In our case $n = 3$.

Taking a Fourier series expansion:

$$a_k = \frac{1}{T}\int_T x(t)\,e^{-jk\frac{2\pi}{T}t} = \frac{1}{T}\int_T x(t)\,e^{-jk\frac{\omega_0}{3}t}\,dt$$

But $x(t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) = \frac{1}{2}\left(e^{j3\frac{\omega_0}{3}t} + e^{-j3\frac{\omega_0}{3}t}\right)$

Then, by the uniqueness of FS coefficients: $\boxed{\begin{array}{l} a_3 = a_{-3} = 1/2 \\ a_k = 0 \quad |k| \neq 3 \end{array}}$

Alternatively, by substitution:

$$a_k = \frac{1}{T}\int_T \frac{1}{2}\left\{ e^{j3\frac{\omega_0}{3}t}\,e^{-jk\frac{\omega_0}{3}t} + e^{-j3\frac{\omega_0}{3}t}\,e^{-jk\frac{\omega_0}{3}t} \right\} dt$$

$$= \frac{1}{2T}\int_T \left(e^{j(3-k)\frac{\omega_0}{3}t} + e^{-j(3+k)\frac{\omega_0}{3}t} \right) dt.$$

Recall that $\displaystyle\int_T e^{j(n-k)\frac{2\pi}{T}t}\,dt = \begin{cases} T & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$

$\Rightarrow$ $a_3 = 1/2$, $a_{-3} = 1/2$ and all other a_k's are zero.

CH. 4 SELECTED HW SOLUTIONS

 # 4.3 a. $x_1(t) = \sin(2\pi t + \pi/4)$

Periodic with fundamental period $T = 1$ $(\omega_0 = 2\pi)$

Fourier Series expansion : $x_1(t) = \frac{1}{2j} e^{j\pi/4} e^{j2\pi t} - \frac{1}{2j} e^{-j\pi/4} e^{-j2\pi t}$

$\therefore$ $a_1 = \frac{1}{2j} e^{j\pi/4}$, $a_{-1} = -\frac{1}{2j} e^{-j\pi/4}$ (the rest are zero)

The Fourier transform of a periodic signal is a train of impulses occuring at $k\omega_0$ $k = 0, \pm 1, \pm 2, \dots$ $\left(\mathcal{F}\{e^{j\omega_0 t}\} = 2\pi\,\delta(\omega - \omega_0) \right)$

So,

$$X_1(j\omega) = 2\pi a_1 \delta(\omega - \omega_0) + 2\pi a_{-1} \delta(\omega + \omega_0)$$
$$= \frac{\pi}{j} e^{j\pi/4} \delta(\omega - 2\pi) + \frac{-\pi}{j} e^{-j\pi/4} \delta(\omega + 2\pi)$$

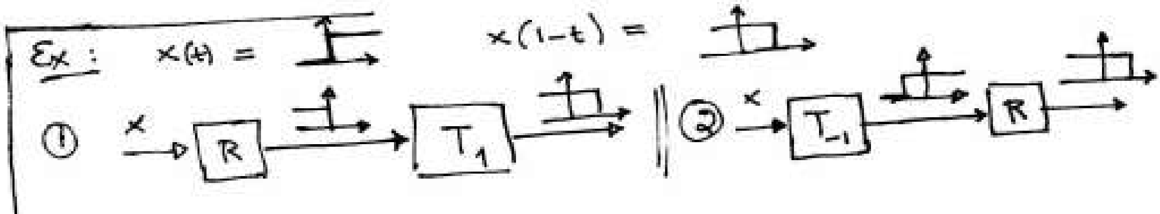
b. $x_2(t) = 1 + \cos(6\pi t + \pi/8)$

With similar arguments,

$$X_2(j\omega) = 2\pi\,\delta(\omega) + \pi e^{j\pi/8} \delta(\omega - 6\pi) + \pi e^{-j\pi/8} \delta(\omega + 6\pi)$$

Notice : $\underset{\uparrow \text{ const. function}}{\mathcal{F}\{1\}} = 2\pi\,\delta(\omega)$

 # 4.6 a. $x(1-t) = T_1 R[x](t) = R T_{-1}[x](t)$



In these problems it is often more convenient to have the translation at the end of the sequence of operations, i.e., $T_{t_0}[\cdots]$

Next, using $\mathcal{F}$-properties,

$$\mathcal{F}\{x(1-t)\} = \mathcal{F}\{T_1 R[x]\} \underset{\text{shift}}{=} e^{-j\omega 1}\,\mathcal{F}\{R[x]\}$$

$$= e^{-j\omega} X(-j\omega)$$

$$\mathcal{F}\{x(-1-t)\} \overset{\text{reflection}}{=} \mathcal{F}\{T_{-1} R[x]\} = e^{j\omega}\,\mathcal{F}\{R[x]\}$$

$$= e^{j\omega} X(-j\omega)$$

Thus, $\mathcal{F}\{x(1-t)+x(-1-t)\} = (e^{j\omega}+e^{-j\omega})\,X(-j\omega)$

$$= \boxed{2\cos\omega\, X(-j\omega)}$$

b. S_a: scaling operation : $S_a[x](t) = x(at)$

$$x(3t-6) = x(3(t-2)) = T_2 S_3 [x](t)$$

$$\Rightarrow \mathcal{F}\{x(3t-6)\} = \mathcal{F}\{T_2 S_3 [x]\} = e^{-2j\omega}\,\mathcal{F}\{S_3[x]\}$$

$$= e^{-2j\omega}\frac{1}{|3|} X\left(\frac{j\omega}{3}\right) = \boxed{\frac{e^{-2j\omega}}{3} X\left(\frac{j\omega}{3}\right)}$$

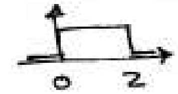
c. $$\mathcal{F}\left\{\frac{d^2}{dt^2}x(t-1)\right\} = \mathcal{F}\left\{T_1 \frac{d^2x}{dt^2}\right\} = e^{-j\omega}\,\mathcal{F}\left\{\frac{d^2x}{dt^2}\right\}$$

$\frac{d}{dt}$ is time-invariant

$$= e^{-j\omega}(j\omega)^2 X(j\omega) = \boxed{-\omega^2 e^{-j\omega} X(j\omega)}$$

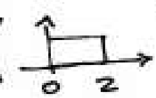
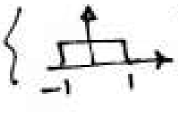
#4.13

$$X(j\omega) = \delta(\omega) + \delta(\omega-\pi) + \delta(\omega-5)$$

$$h(t) = u(t) - u(t-2) =$$ 

a. $x(t)$ is a sum of periodic signals $\left(\frac{1}{2\pi}, \frac{e^{j\pi t}}{2\pi}, \frac{e^{j5t}}{2\pi}\right)$ but the ratio of their frequencies is not a rational number $(\pi/5)$. Hence, $x(t)$ is <u>not</u> periodic.

b. $y = h * x$ is easier to compute in the frequency domain.

$$H(j\omega) = \mathcal{F}\{$$  $$\} = e^{-j\omega}\,\mathcal{F}\{$$  $$\}$$

$$\underset{\text{Tables}}{=} e^{-j\omega}\frac{2\sin\omega}{\omega}$$

$$\Rightarrow Y(j\omega) = H(j\omega)X(j\omega) = e^{-j\omega}\frac{2\sin\omega}{\omega}\left[\delta(\omega)+\delta(\omega-\pi)+\delta(\omega-5)\right]$$

$$= e^{-j0}\,2\left(\frac{\sin\omega}{\omega}\right)\Big|_{\omega\to 0}\delta(\omega) + e^{-j\pi}\frac{2\cancel{\sin\pi}}{\pi}\delta(\omega-\pi) + e^{-j5}\frac{2\sin 5}{5}\delta(\omega-5)$$

0

$\lim_{\omega\to 0}\cos\omega = 1$

$$= 2\delta(\omega) + \frac{e^{-j5}\,2\sin 5}{5}\,\delta(\omega-5)$$

$\therefore$ $y(t)$ is constant + periodic (freq. $= 5$)

$\Rightarrow$ $y(t)$ is periodic.

(In fact $y(t) = \frac{1}{\pi} + \frac{e^{-j5}\sin 5}{5\pi}e^{j5t}$)

c. In (b), neither $x(t)$ nor $h(t)$ were periodic, but their convolution was. Therefore, convolution of aperiodic signals can be periodic.

Note: By means of the Fourier Series Expansion, the Fourier transform of periodic signals should be of the form

$$X(j\omega) = \sum_k a_k \delta(\omega - k\omega_0)$$

for some ω_0. (And vice-versa)

#4.24 Translating the conditions in the time-domain:

1) $\mathrm{Re}\, X(j\omega) = 0 \iff$ Real-Odd signal

2) $\mathrm{Im}\, X(j\omega) = 0 \iff$ Real-even signal

3) $\exists\, \alpha : e^{j\alpha\omega} X(j\omega)$ real $\iff x(t+\alpha)$ real-even signal

4) $\int_{-\infty}^{\infty} X(j\omega)\, d\omega = 0 \iff x(0) = 0 \quad \left(= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega 0} d\omega\right)$

5) $\int_{-\infty}^{\infty} \omega X(j\omega)\, d\omega = 0 \iff \frac{dx}{dt}(0) = 0 \quad \left(= \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega) e^{j\omega 0} d\omega\right)$

6) $X(j\omega)$ periodic $\iff x(t) = \sum_k a_k \delta(t - kT)$ for some T.

For the last ppty, apply FS on $X(j\omega)$ to write

$$X(j\omega) = \sum_k b_k e^{jkT\omega}, \quad \text{for some } T.$$

Then $x(t) = \mathcal{F}^{-1}\{X(j\omega)\} = \sum_k b_k \mathcal{F}^{-1}\{e^{jkT\omega}\} = \sum b_k \delta(t + kT)$

Now, check the time-domain conditions :

a). 1) real-odd $x(t)$ $\Rightarrow$ 1 is satisfied
2) $x(t)$ not even $\Rightarrow$ 2 is not satisfied
3) $x(t+1)$ real-even $\Rightarrow$ 3 is satisfied
4) $x(0)=0$ $\Rightarrow$ 4 is satisfied
5) $\frac{dx}{dt}(0)=1\neq 0$ $\Rightarrow$ 5 is not satisfied
6) $x(t)\neq \sum a_k\delta(t-kT)$ $\Rightarrow$ 6 is not satisfied

Similarly for the rest :

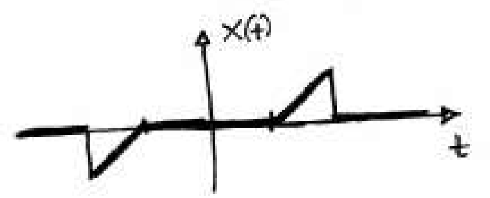
b)		c)	
1	NOT SATISFIED	1	NOT SATISFIED
2	NOT SATISFIED	2	NOT SATISFIED
3	SATISFIED	3	NOT SATISFIED
4	SATISFIED	4	SATISFIED
5	SATISFIED	5	SATISFIED
6	SATISFIED	6	NOT SATISFIED

d)		e)		f)	
1	✓	1	NOT	1	NOT
2	NOT	2	✓	2	✓
3	NOT	3	✓ $(\alpha=0)$	3	✓
4	✓	4	NOT	4	✓
5	NOT	5	✓	5	✓
6	NOT	6	NOT	6	NOT

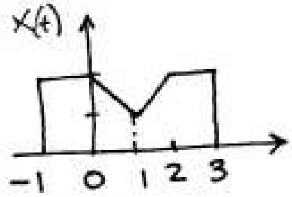
b. A signal that has properties 1, 4, 5 and not 2, 3, 6 :

{Real-odd} & {$x(0)=0$} & {$\frac{dx}{dt}(0)=0$} &

{Not even} & {Not even after shifts} & {Not a sum of equally-spaced impulses}



4.25



a. Look for symmetries after shifting: $x(t+1)$ is even

$\Rightarrow \mathcal{F}\{x(t+1)\}$ is real (angle $= 0$)

$\Rightarrow \mathcal{F}\{x(t)\} = e^{-j\omega}\, \mathcal{F}\{x(t+1)\}$

$\Rightarrow \angle \mathcal{F}\{x(t)\} = -\omega$

b. $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \Rightarrow X(j0) = \int_{-\infty}^{\infty} x(t)\, dt = 2 \cdot 4 - \frac{2 \cdot 1}{2}$
$= \boxed{7}$

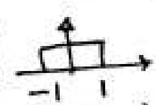
c. $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \Rightarrow \int_{-\infty}^{\infty} X(j\omega)\, d\omega = 2\pi\, x(0)$
$= \boxed{4\pi}$

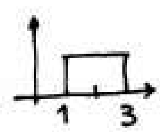
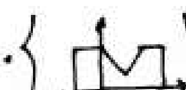
d. Let $Y(j\omega) = X(j\omega)\, \frac{2\sin\omega}{\omega}$

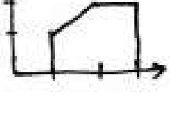
Then $y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\omega)\, e^{j\omega t}\, d\omega$

$\Rightarrow 2\pi y(2) = \frac{\cancel{2\pi}}{\cancel{2\pi}} \int_{-\infty}^{\infty} Y(j\omega)\, e^{2j\omega}\, d\omega$ (the required integral)

But $y(t) = (x(t)) * (\mathcal{F}^{-1}\{\frac{2\sin\omega}{\omega}\})$

$= \{x(t)\} * \{$  $\}$ $= \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$

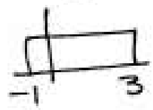
So, $2\pi y(2) = 2\pi \int_{-\infty}^{\infty} h(2-\tau)\, x(\tau)\, d\tau = 2\pi \int_{-\infty}^{\infty}$ $\{$  $\} \cdot \{$  $\}$

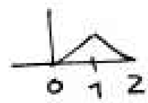
$= 2\pi \int_{-\infty}^{\infty}$ $\{$  $\}$ $= 2\pi\left(2 \cdot 2 - \frac{1}{2}\right) = \boxed{7\pi}$

e. $\displaystyle\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt$ (Parseval)

$\displaystyle = 2\pi \int_{-\infty}^{\infty} x^2(t)\, dt$ (x real)

Let $x(t) = x_0(t) - x_1(t)$





Then, $\int x^2 = \int x_0^2 + \int x_1^2 - 2\int x_0 x_1$

$\displaystyle\int_{-\infty}^{\infty} x_0^2(t)\, dt = 4 \cdot 4 = 16$

$\displaystyle\int_{-\infty}^{\infty} x_0(t)\, x_1(t)\, dt = 2\, \frac{2 \cdot 1}{2} = 2$

$\displaystyle\int_{-\infty}^{\infty} x_1^2(t)\, dt = 2 \int_0^1 t^2 dt = 2\, \frac{1}{3}\, t^3 \Big|_0^1 = 2/3$

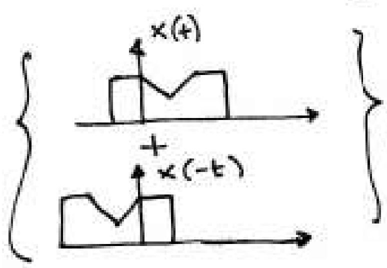
$\Rightarrow \displaystyle\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \boxed{25.333\ \pi}$

f. From the even-odd decomposition property,

$\mathcal{F}^{-1}\{\mathcal{R}e\, X(j\omega)\} = x_e(t) = \mathcal{E}v\{x(t)\}$ (the even part of $x(t)$)

$\displaystyle = \frac{x(t) + x(-t)}{2}$

$=$



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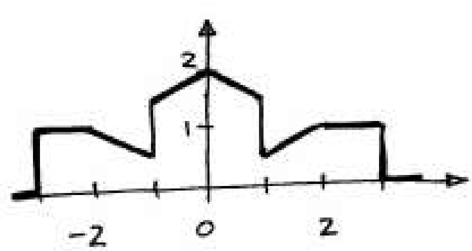


4.26

i) $$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{(2+j\omega)^2}\frac{1}{(4+j\omega)}$$

$$\overset{PFE}{=} \frac{1/4}{4+j\omega} + \frac{-1/4}{2+j\omega} + \frac{1/2}{2+j\omega}$$

Taking $\mathcal{F}^{-1}$,

$$y(t) = \frac{1}{4}e^{-4t}u(t) - \frac{1}{4}e^{-2t}u(t) + \frac{1}{2}te^{-2t}u(t)$$

ii) $$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{(2+j\omega)^2}\frac{1}{(4+j\omega)^2} = \frac{1/4}{2+j\omega} + \frac{1/4}{(2+j\omega)^2} + \frac{-1/4}{(4+j\omega)} + \frac{1/4}{(4+j\omega)^2}$$

$$\overset{\mathcal{F}^{-1}}{\Longrightarrow} y(t) = \frac{1}{4}e^{-2t}u(t) + \frac{1}{4}te^{-2t}u(t) - \frac{1}{4}e^{-4t}u(t) + \frac{1}{4}te^{-4t}u(t).$$

iii) $$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{(1+j\omega)}\frac{1}{(1-j\omega)} = \frac{1/2}{1+j\omega} + \frac{1/2}{1-j\omega}$$

$$\overset{\mathcal{F}^{-1}}{\Longrightarrow} y(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{t}u(-t) = \frac{1}{2}e^{-|t|}$$

b) By direct convolution of $x(t)$ and $h(t)$, $y(t) = \begin{cases} 0 & t<1 \\ 1-e^{-(t-1)} & 1<t\le 5 \\ e^{-(t-5)}-e^{-(t-1)} & t>5 \end{cases}$

$$\overset{\mathcal{F}}{\Longrightarrow} \dots Y(j\omega) = \frac{2e^{-j3\omega}\sin 2\omega}{\omega(1+j\omega)} = \frac{e^{-j2\omega}}{(1+j\omega)}\frac{e^{-j\omega}2\sin 2\omega}{\omega}$$

$$= X(j\omega)H(j\omega)$$

#4.31

$y(t) = \{h(t)\} * \{x(t)\}$ | $x(t) = \cos t$

$\Rightarrow Y(j\omega) = H(j\omega)\, X(j\omega)$ | $X(j\omega) = \pi\left[\delta(\omega-1)+\delta(\omega+1)\right]$

1. $h_1(t) \longrightarrow H_1(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$

$$\Rightarrow Y_1(j\omega) = H_1(j\omega)X(j\omega) = \left[\frac{1}{j\omega}+\pi\delta(\omega)\right]\left[\delta(\omega-1)+\delta(\omega+1)\right]\pi$$

$$= \pi\left(\frac{1}{j}\delta(\omega-1) + \frac{1}{-j}\delta(\omega+1)\right)$$

<u>NOTE</u> $\delta(\omega)\,\delta(\omega-1) = 0$

2. $h_2(t) \rightarrow H_2(j\omega) = -2 + 5\frac{1}{2+j\omega} = \frac{1-2j\omega}{2+j\omega}$

$$\Rightarrow Y_2(j\omega) = \pi\left(\frac{1-2j}{2+j}\delta(\omega-1) + \frac{1+2j}{2-j}\delta(\omega+1)\right)$$

3. $h_3(t) \rightarrow H_3(j\omega) = \frac{2}{(1+j\omega)^2}$

$$\Rightarrow Y_3(j\omega) = \pi\left(\frac{2}{(1+j)^2}\delta(\omega-1) + \frac{2}{(1-j)^2}\delta(\omega+1)\right)$$

Performing the complex computations of the δ-coefficients

$$Y_1(j\omega) = Y_2(j\omega) = Y_3(j\omega).$$

---

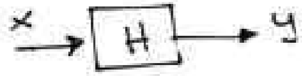
b) The only conditions are $H(j1) = \frac{1}{j}$, $H(-j1) = -\frac{1}{j}$

Let us try a delay, $h(t) = \delta(t-t_0)$ (t_0 TBD)

$\rightarrow H(j\omega) = e^{-j\omega t_0} = \cos\omega t_0 + j\sin\omega t_0 \underset{H(j)=1/j}{\Longrightarrow} t_0 = -\pi/2$

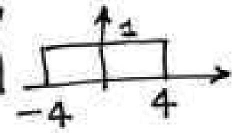
Then $H(-j) = j$ (h real $\Rightarrow$ conjugate symmetry)

4.32



Impulse response of H: $h(t) = \frac{\sin 4(t-1)}{\pi(t-1)}$

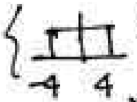
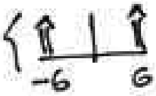
$\Rightarrow h(t) = T_1\left[\frac{\sin 4t}{\pi t}\right]$

$\xrightarrow{\mathcal{F}} H(j\omega) = e^{-j\omega}\Big\{$  $\Big\}$

Next, we compute $X_1(j\omega)$:

$x_1(t) = \cos\left[6(t+\pi/12)\right] = T_{-\pi/12}\left[\cos 6t\right]$

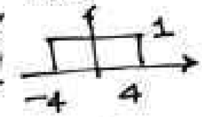
$\xrightarrow{\mathcal{F}} X_1(j\omega) = e^{j\omega\pi/12}\, \pi\left[\delta(\omega-6)+\delta(\omega+6)\right]$

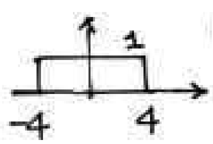
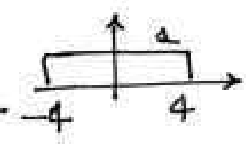
$\Rightarrow Y_1(j\omega) = X_1(j\omega)\, H(j\omega) = \Big\{$  $\Big\} \cdot \Big\{$  $\Big\} \cdot$ constant

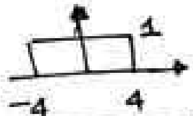
no overlap.

$= 0 \xrightarrow{\mathcal{F}^{-1}} \boxed{y(t) = 0}$

Similarly for $x_3(t)$: $x_3(t) = T_{-1}\left[\frac{\sin 4t}{\pi t}\right]$

$\xrightarrow{\mathcal{F}} X_3(j\omega) = e^{j\omega}\Big\{$  $\Big\}$

$\Rightarrow Y_3(j\omega) = e^{j\omega}\, e^{-j\omega}\Big\{$  $\Big\}\Big\{$  $\Big\}$

$=$ 

$\xrightarrow{\mathcal{F}^{-1}} \boxed{y_3(t) = \frac{\sin 4t}{\pi t}}$

4.33

Transfer function $H(j\omega) = \dfrac{2}{(j\omega)^2 + 6(j\omega) + 8}$

① $h(t) = \mathcal{F}^{-1}\{H(j\omega)\} \overset{PFE}{=} \mathcal{F}^{-1}\left\{\dfrac{A}{j\omega+4} + \dfrac{B}{j\omega+2}\right\}$ $\quad A = \dfrac{2}{s+2}\Big|_{s=-4} = -1$, $\quad B = \dfrac{2}{s+4}\Big|_{s=-2} = 1$

$$= \mathcal{F}^{-1}\left\{\frac{1}{j\omega+2}\right\} - \mathcal{F}^{-1}\left\{\frac{1}{j\omega+4}\right\}$$

$$\overset{TABLES}{=} \boxed{e^{-2t}u(t) - e^{-4t}u(t)}$$

② Either direct convolution or apply $\mathcal{F}$-ppties and take $\mathcal{F}^{-1}$.

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{2}{(j\omega+2)(j\omega+4)}\frac{1}{(j\omega+2)^2}$$

$$\overset{PFE}{=} \frac{A}{j\omega+4} + \frac{B_1}{j\omega+2} + \frac{B_2}{(j\omega+2)^2} + \frac{B_3}{(j\omega+2)^3}$$

$$A = \frac{2}{(s+2)^3}\Big|_{s=-4} = -1/4$$

$$B_3 = \frac{2}{s+4}\Big|_{s=-2} = 1$$

$$B_2 = \frac{d}{ds}\left(\frac{2}{s+4}\right)\Big|_{s=-2} = -1/2$$

$$B_3 = \frac{1}{2!}\frac{d^2}{ds^2}\left(\frac{2}{s+4}\right)\Big|_{s=-2} = 1/4$$

$$\Rightarrow \boxed{y(t) \overset{TABLES}{=} -\frac{1}{4}e^{-4t}u(t) + \frac{1}{4}e^{-2t}u(t) - \frac{1}{2}te^{-2t}u(t) + \frac{t^2}{2}e^{-2t}u(t)}$$

3) $H(j\omega) = \dfrac{2(j\omega)^2 - 2}{(j\omega)^2 + \sqrt{2}\, j\omega + 1} \longleftarrow$ roots $-\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}$ (distinct)

$\deg(\text{numerator}) = \deg(\text{denominator})$

$\Rightarrow H(j\omega) = A + \dfrac{(\text{TBD})}{\text{DENOM.}}$

$\hookrightarrow$ leading coeff. of NUMER. $= 2$

$= 2 + \dfrac{X}{(j\omega)^2 + \sqrt{2}\, j\omega + 1} \quad \underset{\text{solve for } X}{\Rightarrow} \quad 2(j\omega)^2 + 2\sqrt{2}\, j\omega + 2 + X = 2(j\omega)^2 - 2$

$\Rightarrow X = -2\sqrt{2}\, j\omega - 4$

Perform PFE on the last term $:\longrightarrow H(j\omega) = 2 + \dfrac{B}{j\omega + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}} + \dfrac{C}{j\omega + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}}$

$$B = \left.\frac{-2\sqrt{2}\, s - 4}{s + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}}\right|_{s = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}} = \frac{2j - 2}{-j\sqrt{2}} = -\sqrt{2}(1+j)$$

$$C = \left.\frac{-2\sqrt{2}\, s - 4}{s + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}}\right|_{s = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}} = \cdots = B^* = -\sqrt{2}(1-j)$$

since $H(s)$ has real coefficients.

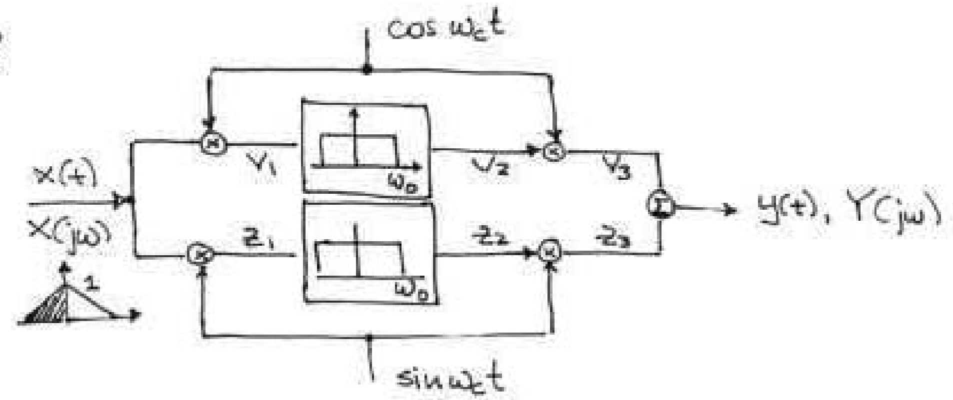
Computing $\mathcal{F}^{-1}$: $\mathcal{F}^{-1}\left\{\frac{B}{\cdots}\right\} = B\, e^{j\left(\frac{-1}{\sqrt{2}}\right)t} e^{-\frac{1}{\sqrt{2}}t} u(t)$

$\mathcal{F}^{-1}\left\{\frac{C}{\cdots}\right\} = (\quad\downarrow\quad)^*$ (complex conjugate)

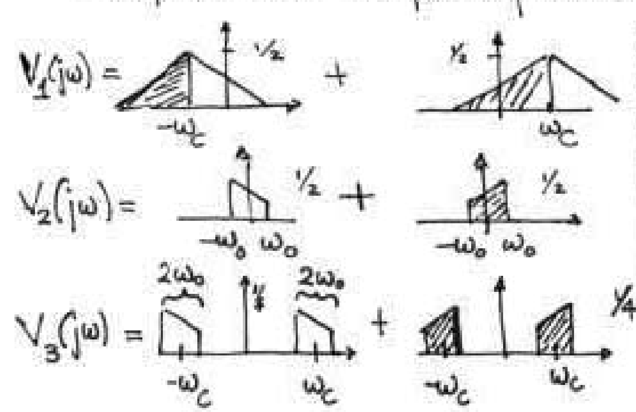
$\mathcal{F}^{-1}\{2\} = 2\delta(t)$

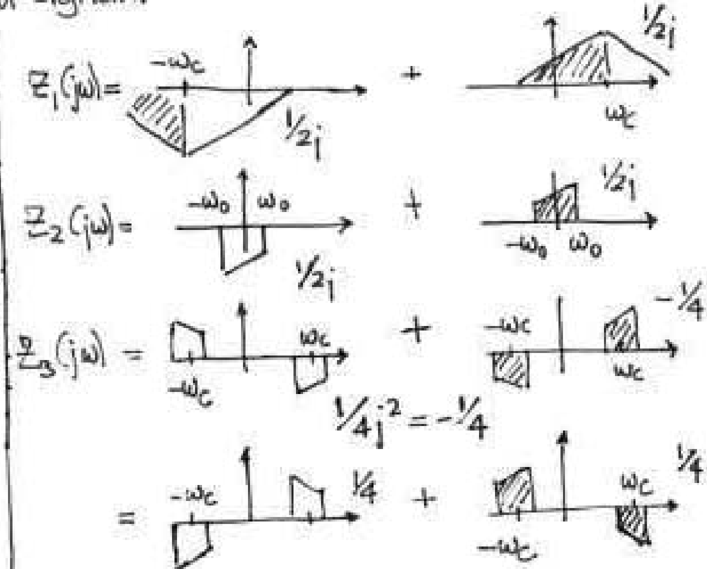
$$\Rightarrow \underline{h(t) = 2\delta(t) + \left\{-2\sqrt{2}\cos(t/\sqrt{2}) - 2\sqrt{2}\sin(t/\sqrt{2})\right\} e^{-t/\sqrt{2}}\, u(t)}$$

#4.46

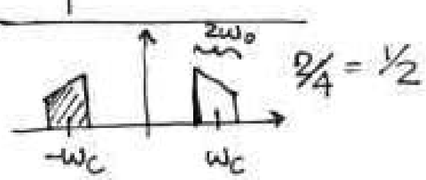


Compute the frequency domain signals:

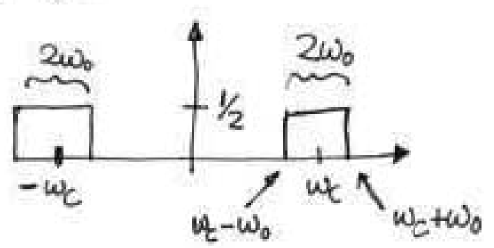




$Y(j\omega) = V_3(j\omega) + Z_3(j\omega) =$



$\therefore$ The system $x(t) \to y(t)$ is equivalent to a bandpass filter with transfer function



$(\omega_c > \omega_0)$

CH 7, 8 SELECTED HW SOLUTIONS

Pr 7.6 Consider the signal $w(t) = x_1(t)\,x_2(t)$. The Fourier transform of $w(t)$, say $W(j\omega)$, is given by

$$W(j\omega) = \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$$

Since $X_1(j\omega) = 0$ for $|\omega| \geq \omega_1$ and $X_2(j\omega) = 0$ for $|\omega| \geq \omega_2$ their convolution will be identically zero for $|\omega| \geq \omega_1 + \omega_2$. Consequently the Nyquist rate for $w(t)$ is $\omega_s = 2(\omega_1 + \omega_2)$. The maximum sampling period that allows complete recovery of $w(t)$ is $T = \frac{2\pi}{\omega_s} = \frac{\pi}{\omega_1 + \omega_2}$.

Pr 7.24 We may express $s(t)$ as $s(t) = \hat{s}(t) - 1$, where $\hat{s}(t)$ is shown in Fig.1 below. Using the tables we may easily show that

$$\hat{S}(j\omega) = \sum_{k=-\infty}^{\infty} \frac{4 \sin 2\pi k \Delta / T}{k}\, \delta(\omega - k\, 2\pi/T)$$

From this, we obtain

$$S(j\omega) = \hat{S}(j\omega) - 2\pi\delta(\omega) = \sum_{k \neq 0} \frac{4 \sin 2\pi k \Delta / T}{k}\, \delta\left(\omega - \frac{2\pi k}{T}\right) + 2\pi\left(\frac{4\Delta}{T} - 1\right)\delta(\omega)$$

Fig. 1

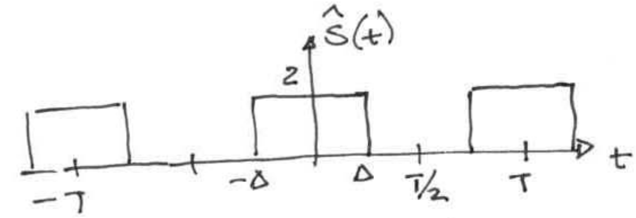
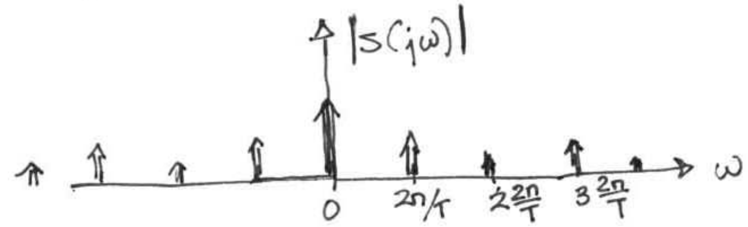


Fig. 2



Thus, $S(j\omega)$ consists of impulses, spaced by $2\pi/T$.

Since $w(t) = s(t)\,x(t)$, $W(j\omega) = \frac{1}{2\pi} S(j\omega) X(j\omega)$ and therefore

$$W(j\omega) = \left(\frac{4\Delta}{T} - 1\right) X(j\omega) + \sum_{k \neq 0} \frac{2\sin(2\pi k \Delta / T)}{k\pi} X\left(j\omega - \frac{2\pi k}{T}\right)$$

That is, $W(j\omega)$ consists of scaled replicas of $X(j\omega)$, spaced by $2\pi/T$. To avoid aliasing, there should be no overlap between any two of these replicas.

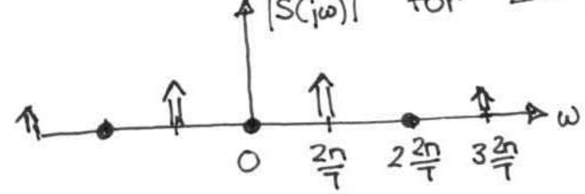
a) $\Delta = T/3$: There are replicas of $X(j\omega)$ centered at $0, 2\pi/T, \dots$, each one having bandwidth ω_M.
To avoid overlaps and aliasing, we should have

$$2\omega_M < \frac{2\pi}{T} \Rightarrow T_{max} = \frac{\pi}{\omega_M}$$

b) $\Delta = T/4$: The same principle holds, except that for this special case, the scaling coefficients for the replicas are $\frac{2\sin 2\pi k \frac{1}{4}}{k\pi} = 0$ for k even

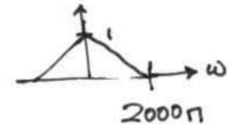
and $\frac{4\Delta}{T} - 1 = 0$, so $X(j\omega)$ does not contribute to $W(j\omega)$.

Fig 3 :

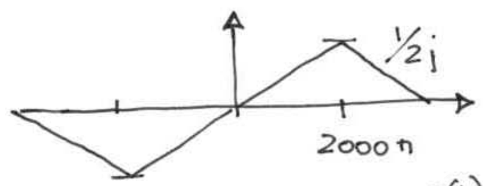


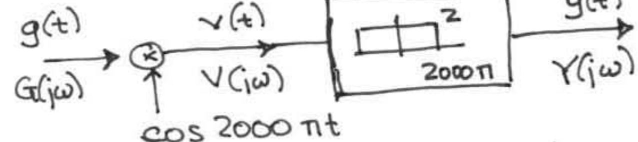
Now, the scaled replicas are spaced by $2\frac{2\pi}{T}$ and the no-overlap condition becomes $2\omega_M < \frac{4\pi}{T} \Rightarrow T_{MAX} = \frac{2\pi}{\omega_M}$

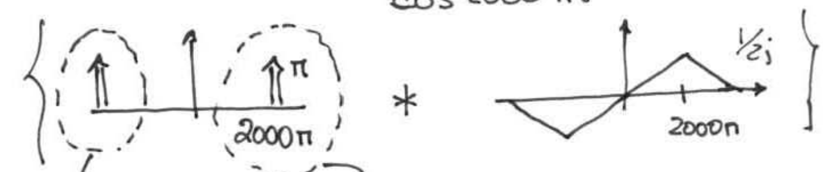
Pr 8.3 We perform the analysis by translating the various operations to the frequency domain.

Let $X(j\omega) =$  (as usual, the exact shape is irrelevant)

$G(j\omega) = \frac{1}{2\pi}$ {  }

= 

The proposed technique is 

So, $V(j\omega) = \frac{1}{2\pi}$ {  }

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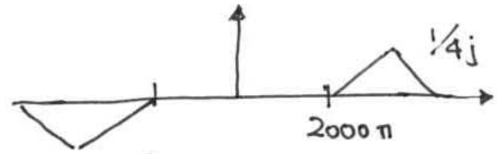
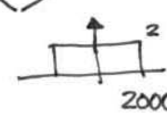
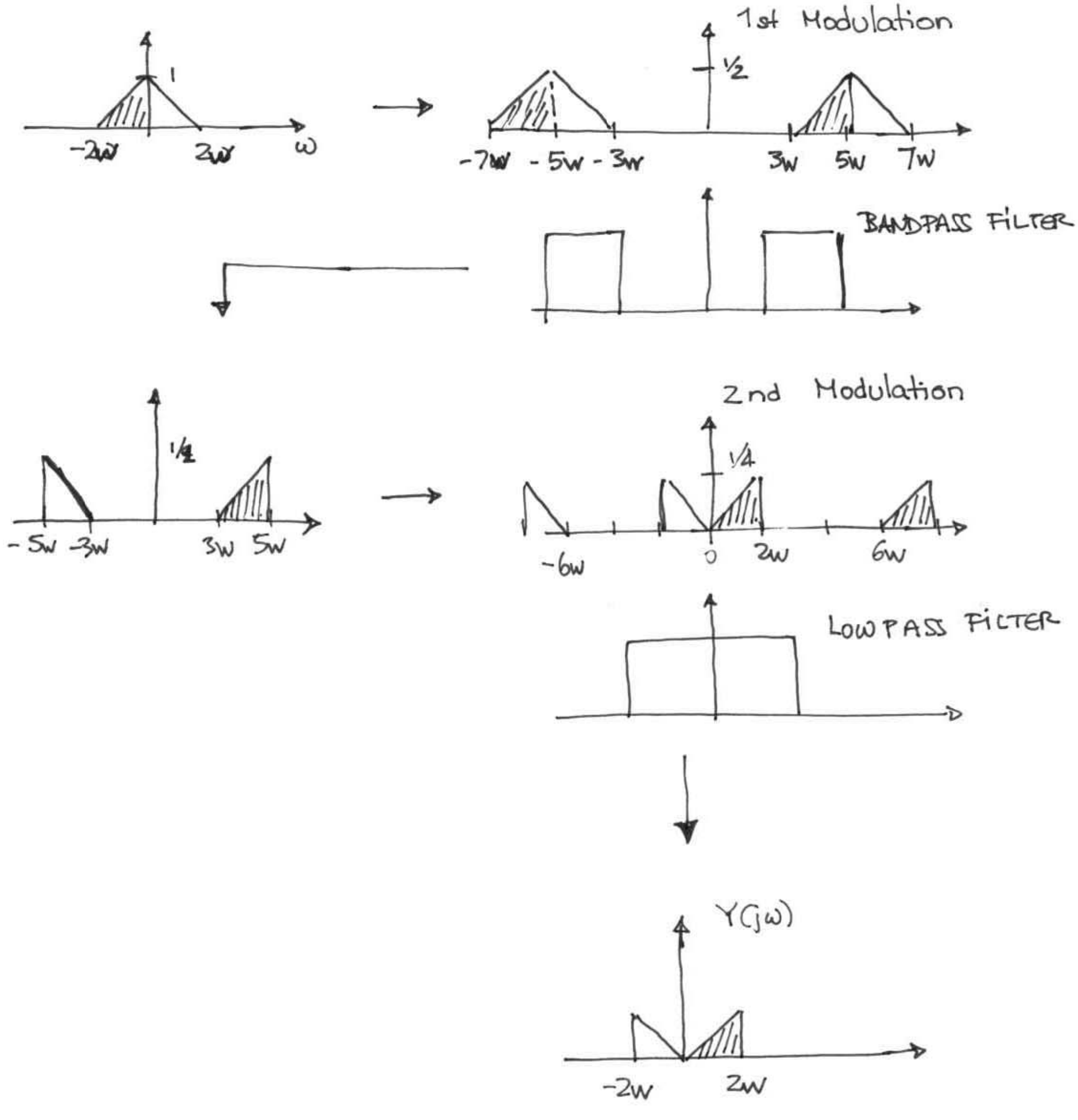
= 

Then $Y(j\omega) =$  $\cdot\ V(j\omega) = 0\,(!) \Rightarrow y(t) = 0$

This technique will not work!

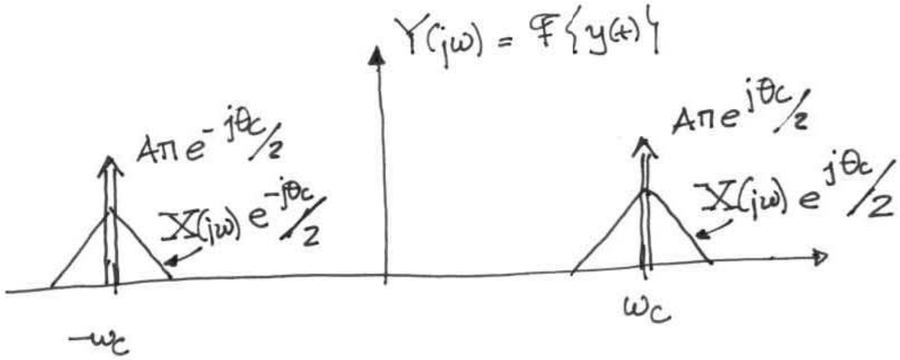
(Compare this result with the use of trigonometric identities to analyze $v(t) = x(t)\sin(2000\pi t)\cos(2000\pi t)$ and then low-pass-filter the result...)

Pr 8.22

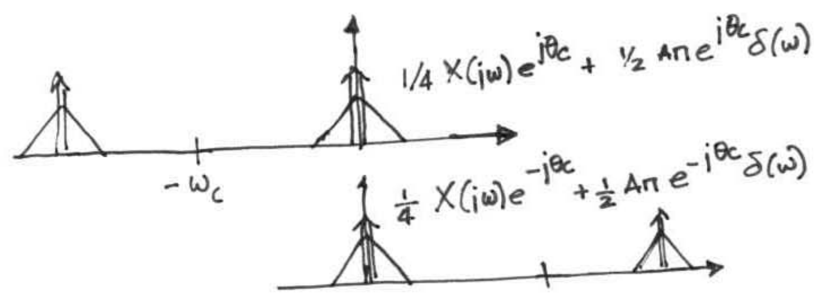


1
-2w
2w
ω
1st Modulation
½
-7w
-5w
-3w
3w
5w
7w
BANDPASS FILTER
¼
-5w
-3w
3w
5w
2nd Modulation
¼
-6w
0
2w
6w
LOWPASS FILTER
Y(jω)
-2w
2w

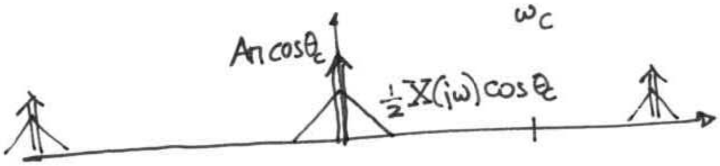
Pr 8.26



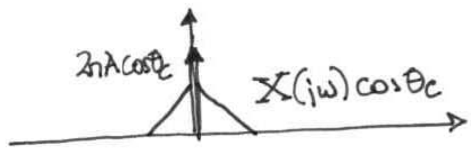
$\mathcal{F}\{y(t) \cos\omega_c t\} =$



$=$



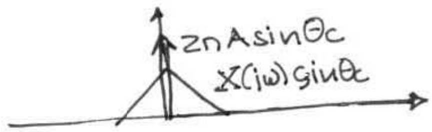
After lowpass $\longrightarrow$



$= \mathcal{F}\{(x(t)+A)\cos\theta_c\}$

Similarly for the $\sin\omega_c t$ modulation, (the side lobes are different)

$\longrightarrow \cdots$ after lowpass $\longrightarrow$



$= \mathcal{F}\{(x(t)+A)\sin\theta_c\}$

Hence, the outputs of the low-pass filters are $(x(t)+A)\cos\theta_c$ and $(x(t)+A)\sin\theta_c$. Squaring and adding we obtain $[x(t)+A]^2[\cos^2\theta_c + \sin^2\theta_c]$ $= [x(t)+A]^2$. Therefore, $r(t) = x(t) + A$. (Note: $x(t)+A > 0$)