EEE 304 **Test 1** **NAME: ___solutions__**

Problem 1:

For the continuous time system with transfer function $H(s) = \dfrac{(s)}{(s+1)(s-2)}$

1. Find the region of convergence of H(s) corresponding to:
   1.1. a stable system.
   1.2. a causal system.
2. Compute the unit step response (x(t)=u(t)), assuming that it is causal.

1. $Stability : ROC = \{-1 < \operatorname{Re} s < 2\}, \quad Causality : ROC = \{\operatorname{Re} s > 2\}$

2. $L\{x\} = \dfrac{1}{s}, ROC = \{\operatorname{Re} s > 0\};$

$$Y(s) = H(s)X(s) = \frac{s}{(s+1)(s-2)}\frac{1}{s} = \frac{1}{(s+1)(s-2)}, ROC = \{\operatorname{Re} s > 2\}$$

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\} = L^{-1}\left\{\frac{-1/3}{s+1} + \frac{1/3}{s-2}\right\}$$

$$= L^{-1}\left\{\frac{-1/3}{s+1}\right\}_{RS} + L^{-1}\left\{\frac{1/3}{s-2}\right\}_{RS} = \left\{\frac{-1}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)\right\}$$

Problem 2:

For the discrete time system with transfer function $H(z) = \dfrac{z-1}{(z-2)(z+2)}$

1. Find the region of convergence of H(z) corresponding to:
   1.1. a stable system.
   1.2. a causal system.
2. Compute the unit step response (x(n)=u(n)) of the system assuming that it is stable.

1. $Stability \Rightarrow ROC_H = \{|z| < 2\}. \quad Causality \Rightarrow ROC_H = \{2 < |z|\}$

2. $Y(z) = H(z)X(z) = \dfrac{(z-1)z}{(z-2)(z+2)(z-1)} = \dfrac{z}{(z-2)(z+2)}; ROC \supseteq \{|z| < 2\} \cap \{1 < |z|\}$

$$= \left\{\frac{1/2}{z-2}\right\}_{LS} + \left\{\frac{1/2}{z+2}\right\}_{LS}$$

$$y(n) = -\frac{1}{2}2^n u(-n-1) - \frac{1}{2}(-2)^n u(-n-1)\Big|_{n \leftarrow n-1} = -\frac{1}{2}2^{n-1}u(-n) - \frac{1}{2}(-2)^{n-1}u(-n)$$

EEE 304 Test 2 Name: ____________________

Problem 1:

For the continuous-time causal system with transfer function $H(s) = \dfrac{(s-20)}{(s+5)}$ compute the following:

1. The amplitude and the phase of the steady-state response to a sinusoid input $x(t) = \sin(10t+30^o)u(t)$.
2. The discrete-time equivalent of H(s), say G(z), using the Forward Euler Approximation and a sampling interval of T = 0.1s.
3. For G(z), compute the amplitude and the phase of the steady-state response to the sinusoid x(t) sampled at the time instants nT, i.e., $x(n) = \sin(n+30^o)u(n)$

1. At steady - state,

$$y(t) = |H(j10)|\sin(10t + 30° + \angle H(j10))$$

$$H(j10) = \frac{(j10-20)}{(j10+5)}$$

$$|H(j10)| = \frac{|(j10-20)|}{|(j10+5)|} = \frac{\sqrt{100+400}}{\sqrt{100+25}} = 2$$

$$\angle H(j10) = \angle(j10-20) - \angle(j10+5) = \tan^{-1}\left(\frac{10}{-20}\right) - \tan^{-1}\left(\frac{10}{5}\right) = 180 - \tan^{-1}(1/2) - \tan^{-1}(2) = 90°$$

2. $$G(z) = H(s)\Big|_{s=\frac{z-1}{T}} = \frac{([z-1]/T - 20)}{([z-1]/T+5)} = \frac{z-1-2}{z-1+0.5} = \frac{z-3}{z-0.5}$$

3. At steady state

$$y(n) = |G(e^{j1})|\sin(n + 30° + \angle G(e^{j1}))$$

$$G(e^{j1}) = \frac{e^{j1}-3}{e^{j1}-0.5} = \frac{[\cos(1)-3] + j\sin(1)}{[\cos(1)-0.5] + j\sin(1)}; \quad \text{Note: angle is in rad so } \cos(1) = 0.54,\ \sin(1) = 0.84$$

$$|G(e^{j1})| = \frac{\sqrt{[\cos(1)-3]^2 + \sin^2(1)}}{\sqrt{[\cos(1)-0.5]^2 + \sin^2(1)}} = 3.09$$

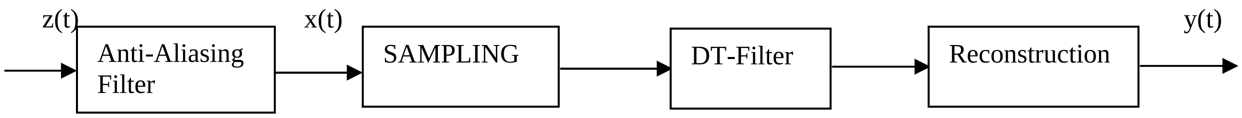
$$\begin{aligned}\angle G(e^{j1}) &= \tan^{-1}(\sin(1)/[\cos(1)-3]) - \tan^{-1}(\sin(1)/[\cos(1)-.5]) + 180; \quad \text{Note: One negative real part} \\ &= 180 + \tan^{-1}(0.84/[0.54-3]) - \tan^{-1}(0.84/[0.54-0.5]) = 180 + \tan^{-1}(-0.34) - \tan^{-1}(21) \\ &= 180 - 18.8 - 87.3 \\ &= 73.9°\end{aligned}$$

EEE 304 **Test 3** **NAME:________________**

Closed-book, closed-notes, calculators and transform tables allowed, 45'

Problem 1:

Suppose that a continuous time signal z(t) has Fourier transform Z(jw) with maximum magnitude 1. The signal is to be converted to discrete time for processing, as shown in the figure below, with sampling time fixed at T = 1ms.



1. Briefly discuss the role of the anti-aliasing filter.
2. Suppose that when |X(jw)| < 0.01, for w larger than the Nyquist frequency, the aliasing effects are negligible. Design a 1st order anti-aliasing filter to meet this specification.
3. Assuming ideal reconstruction, find the frequency response of the above system from x(t) to y(t), when the Discrete-Time Filter has transfer function

$$H(z) = \frac{0.1}{z - 0.9}$$

1. Anti-aliasing filters are analog low-pass filters that are used prior to sampling in order to attenuate the high frequencies of the analog signal and reduce the aliasing effects.
2. A first order anti-aliasing filter has a transfer function of the form $G(s) = \frac{W}{s+W}$, where W is the cutoff frequency. Since X(jw)=G(jw)Z(jw) and |Z(jw)|<1, the specification |X(jw)|<0.01 is satisfied when |G(jw)|<0.01 beyond the Nyquist frequency π/T. But $|G(jw)| = \frac{W}{\sqrt{w^2+W^2}} = \frac{1}{\sqrt{\left(\frac{w}{W}\right)^2+1}}$, so $|G(jw)| \le 0.01 \Rightarrow \left(\frac{w}{W}\right)^2 + 1 \ge 100^2 \xRightarrow{approx} \left(\frac{w}{W}\right) \ge 100$. This inequality should hold for all $w > \frac{\pi}{0.001} \Rightarrow W < 10\pi$.
   Alternatively, a first order filter rolls off at -20dB/dec after its corner frequency (same as bandwidth) and we need -40dB attenuation at Nyquist. Hence, the corner frequency (W) should be 2 decades below Nyquist (1000π).
3. Consider an exponential input $x(t) = e^{jwt}$. After sampling, we get $x_s(n) = e^{jwTn}$. This exponential input produces an exponential output after filtering, i.e., $y_s(n) = H(e^{jwT})e^{jwTn}$. Assuming that the frequency is below Nyquist, the reconstructed signal will be the same frequency, with amplitude as modified by the filter. That is, $y(t) = H(e^{jwT})e^{jwt}$. Hence the frequency response from x to y is $H\left(e^{jwT}\right)$.

Problem 2:
Find the largest sampling interval T_s to allow perfect reconstruction of the signals

1. $\frac{\sin 7t \cos 2t}{t} \overset{Fourier}{===>} \propto\ pulse_{[-7,7]}(w) * [\delta(w-2)+\delta(w+2)] => w_{\max} = 9 => T_s < \frac{\pi}{9}$

2. $\sin t - \sin 3t \overset{F}{=>} w_{\max} = \max(1,3) = 3 => T_s < \frac{\pi}{3}$

3. $e^{-t}u(t) \overset{F}{=>} \frac{1}{jw+1} => w_{\max} = \infty => T_s = 0$ (perfect reconstruction is not possible)

4. $\sin t * \sin 2t \overset{F}{=>} \propto [\delta(w-1)-\delta(w+1)][\delta(w-2)-\delta(w+2)] = 0 => w_{\max} = 0 => T_s \to \infty$
(any Ts will do)

EEE 304 **TEST 4** **NAME:_______________**

Problem:

a. Determine the signal produced if the following sequence of operations is performed on a signal x(t) that is bandlimited to w_m (i.e., X(jw) = 0 for |w|> w_m).

1. Modulation with a square wave carrier of frequency $3w_m$ and an unknown duty cycle "d", i.e.:

$$s(t) = \begin{cases} 1 & |t| < d \\ 0 & otherwise \end{cases} \quad where \quad d \in (0, T/2),\ and\ T\ is\ the\ period\ of\ s(t)$$

2. Bandpass filtering with an ideal filter H(jw) = 1/d for $2w_m$ <|w|< $4w_m$.
3. Modulation with the same square wave carrier.
4. Lowdpass filtering with an ideal filter H(jw) = 1 for |w|< w_m.

b. How does the duty cycle parameter d affect the output signal?

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a.

1. From Fourier tables (or Fourier transform via Fourier Series expansion)

$$S(j\omega) = \sum_k \frac{2\sin kw_0T_1}{k}\delta(\omega - kw_0)\Bigg|_{w_0=3w_m,\,T_1=d} = \sum_k \frac{2\sin k3w_m d}{k}\delta(\omega - k3w_m)$$

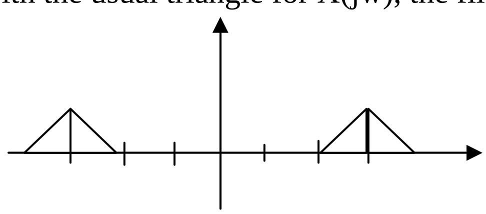
$$X_s(j\omega) = \frac{1}{2\pi}X(j\omega) * S(j\omega) = \frac{1}{2\pi}\sum_k \frac{2\sin k3w_m d}{k} X\left(j(\omega - k3w_m)\right)$$

The replicas of X(jw) are centered at each harmonic frequency $k3w_m$, extending +/- w_m around it.

2. The filtering will allow only the frequencies around the first harmonic to pass. It will eliminate all other components and the DC. The signal amplitude will be the amplitude of the first harmonic multiplied by 1/d (filter) and the original signal X(jw).

$$X_H(j\omega) = H(j\omega)X_s(j\omega) = \begin{cases} \frac{\sin 3w_m d}{\pi d}\left[X\left(j(\omega - 3w_m)\right) + X\left(j(\omega + 3w_m)\right)\right] & |\omega| < 4w_m \\ 0 & otherwise \end{cases}$$

For visualization, with the usual triangle for X(jw), the filtered signal X_H(jw) will be:



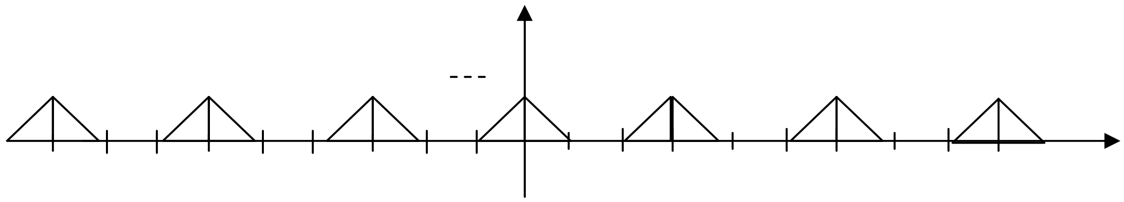
3. Modulating again, we get the same expression as X_s but with X_H in the place of X.

$$X_{Hs}(j\omega) = \frac{1}{2\pi}X_H(j\omega) * S(j\omega) = \frac{1}{2\pi}\sum_k \frac{2\sin k3w_m d}{k} X_H\left(j(\omega - k3w_m)\right)$$

$$= \sum_k \frac{\sin k3w_m d}{\pi k}\frac{\sin 3w_m d}{\pi d}\left[X\left(j(\omega - k3w_m - 3w_m)\right) + X\left(j(\omega - k3w_m + 3w_m)\right)\right]$$

The low frequency signal is obtained for k = 1 and k = -1.

Visualization:



4. After low-pass filtering the low frequency signal is

$$X_{LP}(j\omega) = 2\frac{\sin^2 3w_m d}{\pi^2 d} X(j\omega)$$

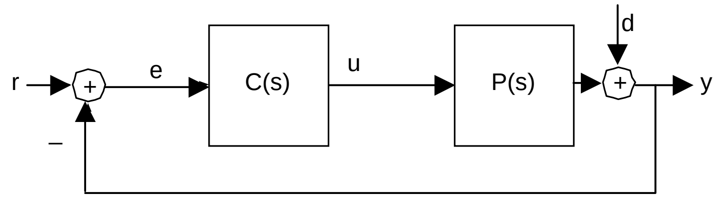
b. The output signal is

$$X_{LP}(j\omega) = 2\frac{\sin^2 3w_m d}{\pi^2 d} X(j\omega)$$

The duty cycle parameter d can take values in (0, T/2) while $3w_m$ is $2\pi/T$. So the sin argument ranges from 0 to π. At the two extremes the numerator of the coefficient of X is zero since in one it is modulated by the zero signal and in the other by a constant. Both get filtered out by the bandpass filter which amplifies the signal by 1/d. The overall coefficient is still zero in both ends but it rises linearly (faster) around 0.
Using numerical evaluation, the peak appears around d = T/5, for which the output amplitude is ~0.9 of the input (X).

EEE 304 **Test 5** **Name: __SOLUTIONS_**

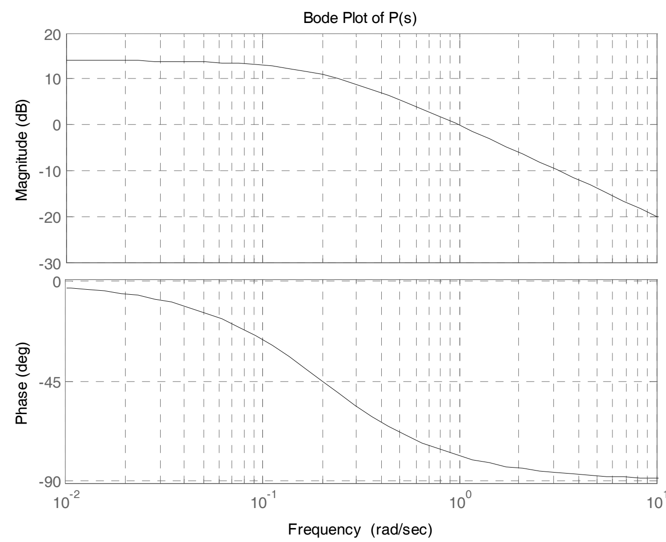
Problem 1:
For the feedback system shown below, compute the transfer functions e/d, u/r



$$\frac{e}{d}=\frac{-1}{1+PC(s)},\quad \frac{u}{r}=\frac{C(s)}{1+PC(s)}$$

Problem 2:
For the feedback system of Problem 1, suppose $P(s)=1/(s+0.2)$ and $C(s)=K(Ts+1)/s$. Determine K,T so that the crossover frequency is 1 and the Phase Margin is at least 60°.

(You may use the given Bode plot to compute the necessary quantities graphically.)



At crossover, the phase condition is

$$\angle\frac{1}{j\omega_c}+\angle K+\angle(Tj\omega_c+1)+\angle\frac{1}{j\omega_c+0.2}=-90^\circ+0^\circ+\tan^{-1}(T\omega_c)-\tan^{-1}(\omega_c/0.2)$$

$$=-180^\circ+60^\circ$$

$$\Rightarrow \tan^{-1}(T)=49^o \Rightarrow T=1.14$$

Using this value we get the gain condition

$$\left|K\frac{Tj\omega_c+1}{j\omega_c}\frac{1}{j\omega_c+0.2}\right|_{\omega_c=1}=K\left.\frac{\sqrt{1.14^2+1}}{\sqrt{1+0.04}}\right|_{\omega_c=1}\Rightarrow K=0.67$$

NAME____Solutions_____

EEE 304 **HW 1**

Problem 1:

Consider the filter with impulse response $h(t) = e^{-4t}u(t)$.

1. Find the transfer function

2. Find the Laplace transform of the output when $x(t) = \sin(5t)u(t)$

3. Find the output by taking the inverse Laplace transform of your answer to part 2.

4. Can you obtain the same result using Fourier Transforms?

1. $H(s) = \dfrac{1}{s+4},\ ROC = \{\mathrm{Re}\, s > -4\}$

2. $L\{x\} = \dfrac{5}{s^2+25},\ ROC = \{\mathrm{Re}\ s > 0\};$

$$Y(s) = H(s)X(s) = \frac{1}{s+4}\frac{5}{s^2+25},\ ROC = \{\mathrm{Re}\ s > 0\}$$

3. $y(t) = L^{-1}\{Y(s)\} = L^{-1}\left\{\dfrac{1}{s+4}\dfrac{5}{s^2+25}\right\} = L^{-1}\left\{\dfrac{A}{s+4} + \dfrac{Bs+C}{s^2+25}\right\}$

$Note: A = 5/[(-4)^2+25] = 5/41;\ B, C: As^2 + 25A + Bs^2 + 4Bs + Cs + 4C = 5 \Rightarrow B = -5/41,\ C = 20/41$

$$= L^{-1}\left\{\frac{5/41}{s+4}\right\} + L^{-1}\left\{\frac{-5/41s}{s^2+25} + \frac{20/41}{s^2+25}\right\} = \left\{\frac{5}{41}e^{-4t}u(t) - \frac{5}{41}\cos(5t)u(t) + \frac{4}{41}\sin(5t)u(t)\right\}$$

4. Yes. But finding the F{sin5t u(t)} is very involved (take the convolution F{sin5t} F{u(t)}) and group terms appropriately. The Fourier approach would fail if the jw-axis is not inside the Laplace ROC nor on its boundary.

Problem 2:

Consider the continuous time causal filter with transfer function

$$H(s)=\frac{1}{(s+1)(s-1)}$$

1. Compute the response of the filter to $x[t] = u[t]$

2. Compute the response of the filter to $x[t] = u[-t]$

3. Repeat parts 1 and 2 for a stable system with the same transfer function.

$$1.\, Causality \Rightarrow ROC_H = \{\text{Re}\, s > 1\}$$

$$Y(s)=H(s)X(s)=\frac{1}{(s+1)(s-1)s};\, ROC \supseteq \{\text{Re}\, s>1\} \cap \{\text{Re}\, s>0\} = \{\text{Re}\, s>1\}$$

$$=\left\{\frac{-1}{s}\right\}_{ROC=\text{Re}\, s>0}+\left\{\frac{1/2}{(s+1)}\right\}_{ROC=\text{Re}\, s>-1}+\left\{\frac{1/2}{(s-1)}\right\}_{ROC=\text{Re}\, s>1}$$

$$y(t)=-u(t)+\frac{1}{2}e^{-t}u(t)+\frac{1}{2}e^{t}u(t)$$

$$2.\, Y(s)=H(s)X(s)=\frac{1}{(s+1)(s-1)(-s)};\, ROC \supseteq \{\text{Re}\, s>1\} \cap \{\text{Re}\, s<0\} = \{\varnothing\}$$

$y(t)$ not well - defined.

$$3.\, Stability \Rightarrow ROC_H = \{-1<\text{Re}\, s<1\}$$

$$3.1: Y(s)=H(s)X(s)=\frac{1}{(s+1)(s-1)s};\, ROC \supseteq \{-1<\text{Re}\, s<1\} \cap \{\text{Re}\, s>0\} = \{0<\text{Re}\, s<1\}$$

$$=\left\{\frac{-1}{s}\right\}_{ROC=\text{Re}\, s>0}+\left\{\frac{1/2}{(s+1)}\right\}_{ROC=\text{Re}\, s>-1}+\left\{\frac{1/2}{(s-1)}\right\}_{ROC=\text{Re}\, s<1}$$

$$y(t)=-u(t)+\frac{1}{2}e^{-t}u(t)+\frac{1}{2}[-e^{t}u(-t)]=-u(t)+\frac{1}{2}e^{-t}u(t)-\frac{1}{2}e^{t}u(-t)$$

$$3.2: Y(s)=H(s)X(s)=\frac{1}{(s+1)(s-1)(-s)};\, ROC \supseteq \{-1<\text{Re}\, s<1\} \cap \{\text{Re}\, s<0\} = \{-1<\text{Re}\, s<0\}$$

$$=-\left\{\frac{-1}{s}\right\}_{ROC=\text{Re}\, s<0}-\left\{\frac{1/2}{(s+1)}\right\}_{ROC=\text{Re}\, s>-1}-\left\{\frac{1/2}{(s-1)}\right\}_{ROC=\text{Re}\, s<1}$$

$$y(t)=-[-u(-t)]-\frac{1}{2}e^{-t}u(t)-\frac{1}{2}[-e^{t}u(-t)]=u(-t)-\frac{1}{2}e^{-t}u(t)+\frac{1}{2}e^{t}u(-t)$$

Problem 3:

Consider the discrete time stable filter with transfer function

$$H(z) = \frac{1}{(z-0.5)(z-2)}$$

1. Compute the response of the filter to $x[n] = u[n]$.

2. Repeat part 1 for a causal filter with the same transfer function.

$$1.\ Stability \Rightarrow ROC_H = \{0.5 <|z|< 2\}$$

$$Y(z) = H(z)X(z) = \frac{z}{(z-0.5)(z-2)(z-1)};\ ROC \supseteq \{0.5 <|z|< 2\} \cap \{1 <|z|\}$$

$$= \left\{\frac{2/3}{z-0.5}\right\}_{ROC=0.5<|z|} + \left\{\frac{4/3}{z-2}\right\}_{ROC=|z|<2} + \left\{\frac{-2}{z-1}\right\}_{ROC=1<|z|}$$

$$y(n) = \frac{2}{3}0.5^{n-1}u(n-1) - \frac{4}{3}2^{n-1}u(-n) - 2u(n-1)$$

$$2.\ Causality \Rightarrow ROC_H = \{2 <|z|\}$$

$$Y(z) = H(z)X(z) = \frac{z}{(z-0.5)(z-2)(z-1)};\ ROC \supseteq \{2 <|z|\} \cap \{1 <|z|\}$$

$$= \left\{\frac{2/3}{z-0.5}\right\}_{ROC=0.5<|z|} + \left\{\frac{4/3}{z-2}\right\}_{ROC=2<|z|} + \left\{\frac{-2}{z-1}\right\}_{ROC=1<|z|}$$

$$y(n) = \frac{2}{3}0.5^{n-1}u(n-1) + \frac{4}{3}2^{n-1}u(n-1) - 2u(n-1)$$

EEE 304 Homework 2

Problem 1:
Consider the following systems:

1. Transfer function $H(s) = \dfrac{s-0.1}{(s+1)(s+10)}$ (Continuous time, causal)

2. Transfer function $H(z) = \dfrac{10(z-1.01)}{(z-0.9)(z)}$ (Discrete time, causal)

Compute the following:

1. Bode plot (expression, graph)
2. Response to sin(2πt) (for CT) and sin(2πn/10) (for DT)

$$1. |H(jw)| = \frac{\sqrt{w^2+0.1^2}}{\sqrt{w^2+1}\sqrt{w^2+10^2}},$$

$$\angle H(jw) = \tan^{-1}(\mathrm{Im}=w, \mathrm{Re}=-0.1) - \tan^{-1}\left(\frac{w}{1}\right) - \tan^{-1}\left(\frac{w}{10}\right) = 180 - \tan^{-1}\left(\frac{w}{0.1}\right) - \tan^{-1}\left(\frac{w}{1}\right) - \tan^{-1}\left(\frac{w}{10}\right)$$

$\Rightarrow$

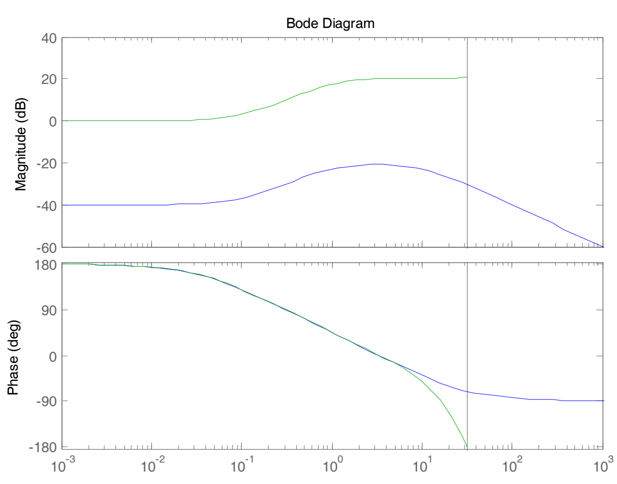
$$y(t) = |H(j6.28)|\sin(t + H(j6.28)) = 0.0836\sin(2\pi t - 22.2°)$$

$$2. |H(e^{j\Omega})| = \frac{10|e^{j\Omega}-1.01|}{|e^{j\Omega}-0.9||e^{j\Omega}|} = \frac{10\sqrt{(\cos\Omega-1.01)^2+(\sin\Omega)^2}}{\sqrt{(\cos\Omega-0.9)^2+(\sin\Omega)^2}},$$

$$\angle H(e^{j\Omega}) = l180 + \tan^{-1}\left(\frac{\sin\Omega}{\cos\Omega-1.01}\right) - \tan^{-1}\left(\frac{\sin\Omega}{\cos\Omega-0.9}\right) - \Omega; \quad l = \#\, Negative\, \mathrm{Re}\, parts$$

$\Rightarrow$

$$y(n) = \left|H\left(e^{j\frac{\pi}{5}}\right)\right| \sin\left(\frac{\pi}{5}n + \angle H\left(e^{j\frac{\pi}{5}}\right)\right) = 10.44\sin\left(\frac{\pi}{5}n - 25.9°\right)$$



NOTES:
- The DT system is a discrete approximation of the continuous one with sampling time T = 0.1. It is also scaled by a gain of 100. The approximation is good up to frequencies ~0.1(Nyquist) = 0.1(3.14/T) = 3.14 rad/s. Thus, the transfer function roll-off at high frequencies does not appear in the DT system. Also, our frequency is beyond this (arbitrary) limit by a factor of two. We xpect some deviation of the DT results from the CT ones (un-scaled magnitude 0.104 vs 0.084, phase 26^o vs 22^o)
- In the computation of magnitude and phase of the DT system the DT frequency Ω = ωT = π/5 rad/sample was used. However, when using MATLAB's "bode" command, the frequency must be converted to rad/sec (ω=2 π). That is,
Hd=tf(10*[1 -1.01],[1 -.9 0],.1) [m,p]=bode(Hd,6.28), yields the correct result m = 10.44, p = -25.9.

Problem 2:

1. Use forward and backward Euler approximations of derivative to derive the DT counterparts of the system with transfer function $H(s) = \dfrac{-0.2s+1}{(s+1)(s+2)}$, for sampling times 0.1, 1.

$$Forward-Euler: s = \frac{z-1}{T} \Rightarrow H_d(z) = \frac{-0.2\dfrac{z-1}{T}+1}{\left(\dfrac{z-1}{T}+1\right)\left(\dfrac{z-1}{T}+2\right)} = \frac{(-0.2z+0.2+T)T}{(z-1+T)(z-1+2T)}$$

$$= \frac{-0.2Tz+(0.2+T)T}{(z-(1-T))(z-(1-2T))}$$

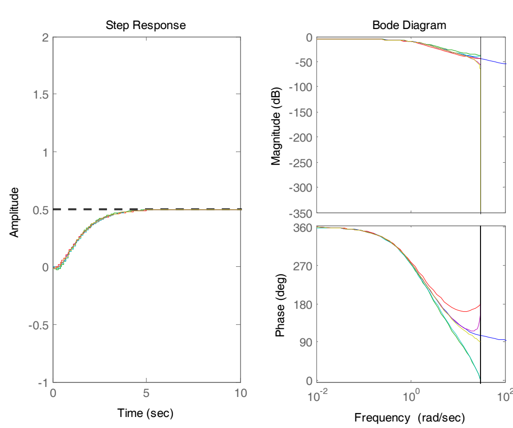
$$Backward-Euler: s = \frac{1-z^{-1}}{T} \Rightarrow H_d(z) = \frac{-0.2\dfrac{z-1}{zT}+1}{\left(\dfrac{z-1}{Tz}+1\right)\left(\dfrac{z-1}{Tz}+2\right)} = \frac{((-0.2z+0.2+Tz)Tz}{(z-1+Tz)(z-1+2Tz)}$$

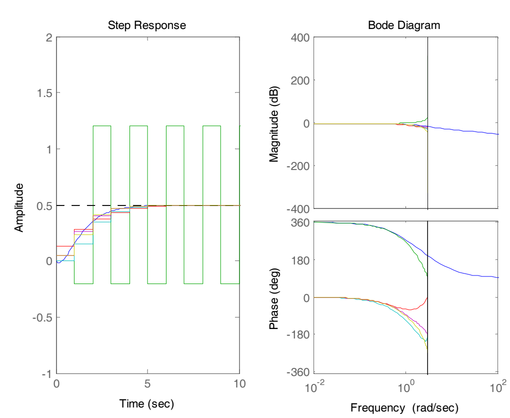
$$= \frac{((T-0.2)z+0.2)Tz}{((1+T)z-1)((1+2T)z-1)}$$

2. Use MATLAB to compare the step responses and frequency responses of the discretizations in P.2.1 with the CT transfer function, and its descretization using the function c2d, with Tustin, zoh and foh options (sample code is given below). Briefly, describe your observations.

- All approximations are good up to one order of magnitude below the Nyquist frequency. Some continue to be reasonable up to 1/3 of Nyquist frequency.
- Forward Euler fails when T = 1 (|T*pole| < 2 is violated).

```
H=tf([-.2 1],[1 3 2]);T=.1
num=T*[T-0.2  0.2 0];den=conv([1+T -1],[1+2*T,-1]);Hdb=tf(num,den,T)
num=[-0.2*T 0.2*T+T*T];den=conv([1 -1+T],[1 -1+2*T]);Hdf=tf(num,den,T)
Hdzoh=c2d(H,T,'zoh'),Hdfoh=c2d(H,T,'foh'),Hdtust=c2d(H,T,'tustin'),
subplot(121)
step(H,Hdf,Hdb,Hdzoh,Hdfoh,Hdtust)
axis([0,10,-1 2])
subplot(122)
bode(H,Hdf,Hdb,Hdzoh,Hdfoh,Hdtust)
```





EEE 304 **HW 3** **SOLUTIONS**

Problem 1:
Do Problems 7.28, 7.31 from the textbook.

7.28
Consider the periodic signal $x(t) = \sum_k a_k e^{jkw_0 t}$, $w_0 = \frac{2\pi}{T} = 20\pi,\ a_k = \frac{1}{2^{|k|}}$. This signal is filtered by an ideal Anti-Aliasing Filter with cutoff-frequency 205π. Hence, only the terms with frequency less than the cutoff will pass (with the same amplitude) and the rest will be rejected. For an exponential to be in the pass band we must have $kw_0 < 205\pi \Rightarrow \mathrm{k} < 205/20 \Rightarrow \mathrm{k} \leq 10$.
So, $x_c(t) = \sum_{k=-10}^{10} a_k e^{jkw_0 t} = \sum_{k=-10}^{10} a_k e^{jk20\pi t}$. When this signal is sampled with sampling time Ts (need a different symbol here; the book uses the same creating confusion)

$$x_c(t)p(t) = \sum_n \delta(t - nTs) \sum_{k=-10}^{10} a_k e^{jkw_0 t} = \sum_n \sum_{k=-10}^{10} a_k e^{jk20\pi nTs}\, \delta(t - nTs)$$
$$= \sum_n \sum_{k=-10}^{10} a_k e^{jk\frac{2\pi}{20}n}\, \delta(t - nTs)$$

Thus, $x(n) = \sum_{k=-10}^{10} a_k e^{jk\frac{2\pi}{20}n}$.

a. This sequence is periodic in n, with the period being N = 20.
b. Furthermore, by comparing directly with (3.94), x(n) is in the form of a Fourier Series expansion (Discrete Time), so the coefficients are simply $a_k = \frac{1}{2^{|k|}}$.

7.31
Consider a test signal $x_c(t)=\exp(jwt)$, with $w < \pi/T$. Then, following the operations in Fig.P.7.31,

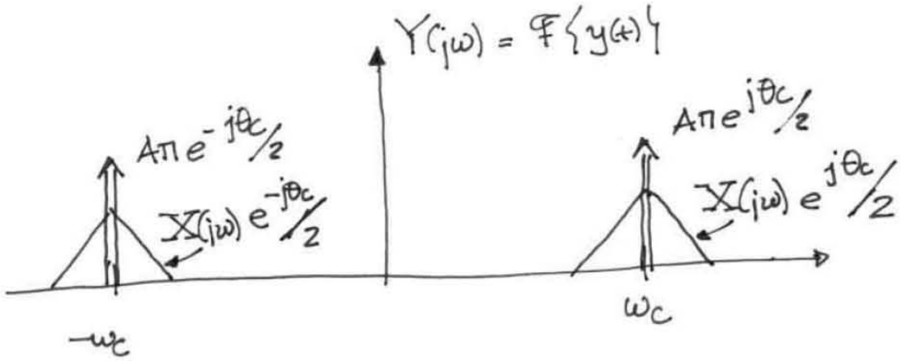
$$x(n) = e^{jwnT} = \left(e^{jwT}\right)^n$$
$$y(n) = \frac{1}{2}y(n-1) + x(n) = H(z)\big|_{z=e^{jwT}} \left(e^{jwT}\right)^n \text{ (since x is an exponential)}$$
$$= \frac{1}{1-\frac{1}{2}z^{-1}}\Bigg|_{z=e^{jwT}} \left(e^{jwT}\right)^n = \frac{2}{2-e^{-jwT}}\left(e^{jwT}\right)^n$$
$$y_c(t) = Lowpass\left[\frac{2}{2-e^{-jwT}}e^{jwnT}\right] = \frac{2}{2-e^{-jwT}}e^{jwt} \quad \text{(because } w < \pi/T\text{)}$$
$$\Rightarrow H(jw) = \frac{2}{2-e^{-jwT}}; \text{ (since, for an LTI system with an exponential input } x_c(t) = e^{jwt},\ y_c(t) = H(jw)e^{jwt}\text{)}$$

Notice that from the last expression, the transfer function is H(s) = 2/(2-exp(-sT)) which is not a finite dimensional system. Since the book does not specify the amplitude of the lowpass, in the reconstruction we assumed that the low-pass has the correct amplitude to recover the signal (i.e., T). If we assume an amplitude of 1, the transfer function must be divided by T.

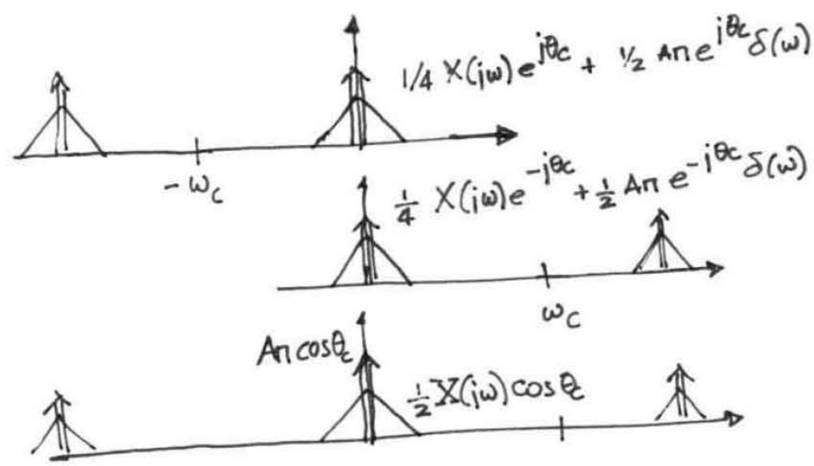
Problem 2:
Find the largest sampling interval T_s to allow perfect reconstruction of the signals (x*y denotes convolution)

1. $\frac{\sin t}{t}\sin 2t \Rightarrow \mathrm{w}_{\max} = 1+2 \Rightarrow T_s = \frac{\pi}{w_{\max}} = \frac{\pi}{3}$ (using convolution of Fourier transforms).
2. $\frac{\sin 2t}{t} * \sin t \Rightarrow \mathrm{w}_{\max} = \min(2,1) \Rightarrow T_s = \frac{\pi}{1}$
3. $\frac{\sin 2t}{t}\frac{\sin 3t}{t} \Rightarrow \mathrm{w}_{\max} = 2+3 \Rightarrow T_s = \frac{\pi}{w_{\max}} = \frac{\pi}{5}$.
4. $\frac{\sin t}{t} * \sin 4t \Rightarrow \mathrm{w}_{\max} \le \min(1,4), \Rightarrow T_s = \frac{\pi}{w_{\max}} = \frac{\pi}{1}$. In fact, if we compute the product of the fourier transforms we get 0 so any Ts > 0 will do.

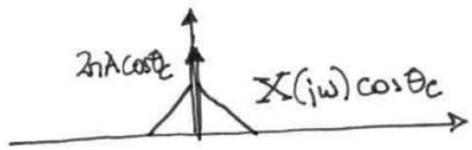
Pr 8.26



$\mathcal{F}\{y(t)\cos\omega_c t\} =$



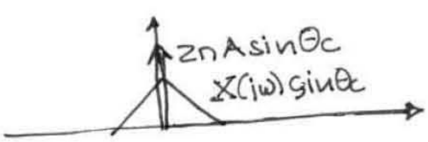
After lowpass $\longrightarrow$



$= \mathcal{F}\{(x(t)+A)\cos\theta_c\}$

Similarly for the $\sin\omega_c t$ modulation, (the side lobes are different)

$\longrightarrow \cdots$ after lowpass $\longrightarrow$

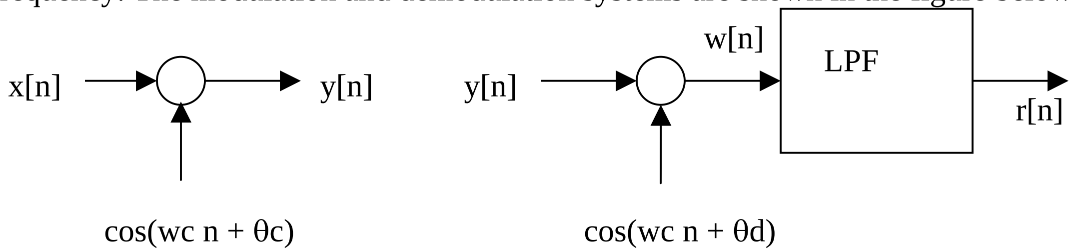


$= \mathcal{F}\{(x(t)+A)\sin\theta_c\}$

Hence, the outputs of the low-pass filters are $(x(t)+A)\cos\theta_c$ and $(x(t)+A)\sin\theta_c$. Squaring and adding we obtain $[x(t)+A]^2[\cos^2\theta_c+\sin^2\theta_c]$ $= [x(t)+A]^2$. Therefore, $r(t) = x(t) + A$. (Note: $x(t)+A > 0$)

Problem 8.47 and Solution

In this problem we want to consider the effect of a loss in synchronization in phase and/or frequency. The modulation and demodulation systems are shown in the figure below.



For parts (a) and (b) of this problem, the difference in frequency is zero, and the difference in the phase is denoted by $\Delta\theta = \theta d - \theta c$.

(a) If the spectrum is shown in Figure P8.47(b), sketch the spectrum of w[n]. (The spectrum is a symmetric triangle that is bandlimited at frequency wm.)
(b) Show that w can be chosen so that the output r[n] is $r[n] = x[n] \cos \Delta\theta$. In particular, what is r[n] if $\Delta\theta = \pi/2$?

Solution

Observe the following

$$y[n] = x[n] \cos (wc\, n + \theta c)$$
$$w[n] = y[n] \cos (wc\, n + \theta d)$$
$$w[n] = x[n] \cos (wc\, n + \theta c) \cos (wc\, n + \theta d)$$
$$w[n] = 0.5\, x[n] \cos \Delta\theta + 0.5\, x[n] \cos (2wc\, n + \theta c + \theta d)$$

Observe that in the frequency domain the first term in the equation for w[n] is a scaled version of the original signal and that the second term is a scaled and shifted version of the original signal. This observation can be used to sketch the result.

There are actually two cases here, depending on the magnitude of the phase shifts. One case has overlap the other does not.

Finally, if the carrier frequency is chosen correctly, it is possible to use an ideal low-pass filter (in the frequency domain) to filter out the second term in the equation for w[n]. In the special case where the phase shift is $\pi/2$, then the output r[n] is zero.

Problem 3: (8.49 of textbook)

(a) s(t) periodic so its Fourier transform is computed through the Fourier series expansion. That is,

$$s(t) = \sum a_k e^{jk\frac{2\pi}{T}t}, \quad \omega_o = \frac{2\pi}{T}$$

$$S(j\omega) = 2\pi \sum a_k \delta\left(\omega - k\frac{2\pi}{T}\right), \quad a_k = \frac{\sin k\omega_o T_1}{k\pi}, \quad T_1 = \frac{T}{4} \quad \Rightarrow a_k = \frac{\sin k\pi/2}{k\pi}$$

$$a_0 = 1/2$$

(The a_k is zero for even k, other than 0.)

The modulated (chopped) x(t) has Fourier transform

$$X_s(j\omega) = \frac{1}{2\pi} X(j\omega) * S(j\omega) = \sum a_k X\left(j\omega - jk\frac{2\pi}{T}\right)$$

yielding the band-pass filtered-chopped signal, say v(t)

$$V(j\omega) = H_1(j\omega) \sum a_k X\left(j\omega - jk\frac{2\pi}{T}\right) = Aa_1 X\left(j\omega - j\frac{2\pi}{T}\right) + Aa_{-1} X\left(j\omega + j\frac{2\pi}{T}\right)$$

$$= \frac{A}{\pi} X\left(j\omega - j\frac{2\pi}{T}\right) + \frac{A}{\pi} X\left(j\omega + j\frac{2\pi}{T}\right)$$

The maximum allowable frequency content in x(t) for this expression to be valid is $\omega_M < \pi/T$.

Next, V(jω) is re-modulated (chopped) and low-pass filtered. The modulation by s(t) produces two replicas of X(jω) at 0, each multiplied by $1/\pi$, yielding a total coefficient of $2A/\pi^2$. The rest of the replicas are at $k2\pi/T$ that are filtered out by the H_2 low-pass filter. So,

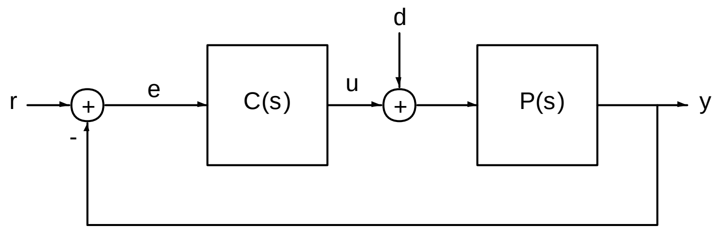
$$Y(j\omega) = H_2(j\omega) V_s(j\omega) = \frac{2A}{\pi^2} X(j\omega)$$

with the same condition on the maximum frequency in x(t), $\omega_M < \pi/T$.

(b) From the last expression, the equivalent gain of the overall system is $2A/\pi^2$.

EEE 304 **HW 5** **SOLUTIONS**

Problem 1:
For the feedback system shown below, compute the transfer functions y/r, y/d, u/r, u/d.



$$\frac{y(s)}{r(s)} = \frac{PC}{1+PC}, \quad \frac{y(s)}{d(s)} = \frac{P}{1+PC}, \quad \frac{u(s)}{r(s)} = \frac{C}{1+CP}, \quad \frac{u(s)}{d(s)} = \frac{-CP}{1+CP},$$

Problem 2: (Low Bandwidth Controller)
For the feedback system of Problem 1, suppose $P(s) = 1/(s+1)$.

a. When *C(s) = K*, design K so that the loop crossover frequency (i.e., w: |P(jw)C(jw) |= 1) is 0.5. What is the contribution of a constant unit disturbance to the output?
b. When $C(s) = K(Ts+1)/s$, design K,T so that the crossover frequency is 0.5 and the phase margin (i.e., the difference between the loop angle and –180 at the crossover frequency, $\angle P(jw_c)C(jw_c)+180$) is at least 50°. What is the contribution of a constant unit disturbance to the output?

a. $\omega_c = 0.5 : |P(j\omega_c)C(j\omega_c)| = 1 \Rightarrow \left|\frac{K}{\sqrt{1+(\omega_c)^2}}\right| = 1 \Rightarrow |K| = \sqrt{1.25} = 1.12$

$K > 0$ for stability ($\angle C + \angle P > -180$)

$y_d(s) = \frac{P}{1+PC}d(s) = \frac{1}{s+K+1}\frac{1}{s} = \frac{-1/2.12}{s+2.12} + \frac{1/2.12}{s} \Rightarrow y_d(t) = \frac{1}{2.12}\left(1-e^{-2.12t}\right)U(t)$

$\Rightarrow y_{d,ss} = \lim_{t\to\infty} y_d(t) = 0.47$

b. $\omega_c = 0.5 : \angle C + \angle P > -180 + 50 = -130$

$\Rightarrow \tan^{-1}(T\omega_c) - 90 - \tan^{-1}(\omega_c) > -130 \Rightarrow T\omega_c > -40 + 26.5 \Rightarrow T = 0$

$\omega_c = 0.5 : |P(j\omega_c)C(j\omega_c)| = 1 \Rightarrow \left|\frac{K}{\omega_c\sqrt{1+(\omega_c)^2}}\right| = 1 \Rightarrow |K| = 0.56 \Rightarrow C(s) = 0.56/s$

$K > 0$ for stability ($\angle C + \angle P > -180$)

$y_d(s) = \frac{P}{1+PC}d(s) = \frac{s}{s^2+s+K}\frac{1}{s} = \frac{1}{s^2+s+0.56}$

$\overbrace{\Rightarrow}^{a=0.5,wo=0.56} \quad y_d(t) = 1.8e^{-0.5t}\sin(0.56t)U(t) \Rightarrow y_{d,ss} = \lim_{t\to\infty} y_d(t) = 0$

Verify in MATLAB, using **step(feedback(P,1.12),feedback(P,C))**

Problem 3: (High Bandwidth Controller)
For the feedback system of Problem 1, suppose $P(s) = 1/(s+1)$.

a. When *C(s) = K*, design K so that the loop crossover frequency (i.e., w: |P(jw)C(jw) |= 1) is 20. What is the contribution of a constant unit disturbance to the output?
b. When $C(s) = K(Ts+1)/s$, design K,T so that the crossover frequency is 20 and the phase margin (i.e., the difference between the loop angle and –180 at the crossover frequency, $\angle P(jw_c)C(jw_c)+180$) is at least 50°. What is the contribution of a constant unit disturbance to the output?

a. $\omega_c = 20: |P(j\omega_c)C(j\omega_c)| = 1 \Rightarrow \left|\dfrac{K}{\sqrt{1+(\omega_c)^2}}\right| = 1 \Rightarrow |K| = \sqrt{401} = 20$

$K > 0$ for stability ($\angle C + \angle P > -180$)

$$y_d(s) = \frac{P}{1+PC}d(s) = \frac{1}{s+K+1}\frac{1}{s} = \frac{-1/21}{s+21} + \frac{1/21}{s} \Rightarrow y_d(t) = 0.048\left(1-e^{-21t}\right)U(t)$$

$\Rightarrow y_{d,ss} = \lim_{t\to\infty} y_d(t) = 0.048$

b. $\omega_c = 20: \angle C + \angle P = -180 + 50 > -130$

$\Rightarrow \tan^{-1}(T\omega_c) > -130 + 90 + 87.1 = 47.1 \Rightarrow T\omega_c = 1.08 \Rightarrow T = 0.054$

$$\omega_c = 20: |P(j\omega_c)C(j\omega_c)| = 1 \Rightarrow \left|\frac{K\sqrt{1+(T\omega_c)^2}}{\omega_c\sqrt{1+(\omega_c)^2}}\right| = 1 \Rightarrow |K| = 272 \Rightarrow C(s) = 272(0.054s+1)/s$$

$K > 0$ for stability ($\angle C + \angle P > -180$)

$$y_d(s) = \frac{P}{1+PC}d(s) = \frac{s}{s^2+s+K(Ts+1)}\frac{1}{s} = \frac{1}{s^2+(1+KT)s+K} = \frac{1}{s^2+15.7s+272}$$

$$\overbrace{\Rightarrow}^{a=7.85, wo=14.5} \quad y_d(t) = 0.07e^{-7.85t}\sin(14.5t)U(t) \Rightarrow y_{d,ss} = \lim_{t\to\infty} y_d(t) = 0$$

Verify in MATLAB, using **step(feedback(P,20),feedback(P,C))**

Problem 4: (Optional, 10% bonus)
Select a suitable sampling rate and use your favorite continuous-to-discrete conversion method to discretize the controllers of P.2 and P.3 (obtain discrete-time "equivalents"). Simulate the responses in SIMULINK (discrete-time controller, continuous time system).

For Pr.3.b, following the rule of 6 samples/rise-time, we have t_r = 2/BW ~ 2/20 = 0.1.
Hence, the sampling rate is 6 samples/0.1sec or 60sam/sec => Ts = 0.017sec.
Using the Forward Euler method (no fast poles=> no sampling constraints), we get the discrete equivalent Cd(z) = (14.69z-10.06)/(z-1). Simulink will accept the connection of DT and CT systems as long as there is only one sampling time. (For multirate systems, rate conversion blocks must be used.) The SIMULINK GUI is shown in the figure below.