**EEE 304** **Test 1** **NAME: ___________________**

**Problem 1:**

For the continuous time system with transfer function $H(s) = \dfrac{1}{(s+0.5)(s+2)}$

1. Find the region of convergence of H(s) corresponding to:
    1.1. a stable system.
    1.2. a causal system.
2. Compute the unit step response (x(t)=u(t)), assuming that it is stable.

$$1.1: ROC = \{-0.5 < \operatorname{Re} s\}$$

$$1.2: ROC = \{-0.5 < \operatorname{Re} s\}$$

$$2: Y(s) = \frac{1}{(s+0.5)(s+2)s} \quad ROC = \{-0.5 < \operatorname{Re} s\} \cap \{0 < \operatorname{Re} s\}$$

$$= \underbrace{\frac{1}{s}}_{causal} + \underbrace{\frac{-4/3}{(s+0.5)}}_{causal} + \underbrace{\frac{1/3}{(s+2)}}_{causal}$$

$$y(t) = u(t) - \frac{4}{3}e^{-0.5t}u(t) + \frac{1}{3}e^{-2t}u(t)$$

**Problem 2:**

For the discrete time system with transfer function $H(z) = \dfrac{1}{(z+0.5)(z+2)}$

1. Find the region of convergence of H(z) corresponding to:
    1.1. a stable system.
    1.2. a causal system.
2. Compute the unit step response (x(n)=u(n)) of the system assuming that it is causal.

$$1.1: ROC = \{0.5 < |z| < 2\}$$

$$1.2: ROC = \{2 < |z|\}$$

$$2: Y(z) = \frac{z}{(z+0.5)(z+2)(z-1)} \quad ROC = \{2 < |z|\} \cap \{1 < |z|\}$$

$$= \underbrace{\frac{2/9}{z-1}}_{causal} + \underbrace{\frac{2/9}{(z+0.5)}}_{causal} + \underbrace{\frac{-4/9}{(z+2)}}_{causal} = z^{-1}\left\{\frac{2}{9}\underbrace{\frac{z}{z-1}}_{causal} + \frac{2}{9}\underbrace{\frac{z}{(z+0.5)}}_{causal} - \frac{4}{9}\underbrace{\frac{z}{(z+2)}}_{causal}\right\}$$

$$y(n) = \frac{2}{9}u(n-1) + \frac{2}{9}(-0.5)^{n-1}u(n-1) - \frac{4}{9}(-2)^{n-1}u(n-1)$$

# NAME___SOLUTIONS____

# EEE 304 HW 2

**Problem 1:**

Consider the filter with impulse response $h(t) = e^{-4t}u(t)$.

1. Find the transfer function

2. Find the Fourier transform of the output when $x(t) = \sin(5t)u(t)$

3. Find the output when $x(t) = \sin(5t)u(t)$ by taking the inverse Fourier transform of your answer to part 2.

4. Repeat parts 2 and 3 using Laplace Transform

1. $H(s) = \frac{1}{s+4}, ROC = \{\mathrm{Re}\, s > -4\}; \quad H(jw) = \frac{1}{jw+4}$

2. $F\{x\} = \frac{1}{2\pi} F\{\sin 5t\} * F\{u(t)\} = \frac{1}{2j}[\delta(w-5) - \delta(w+5)] * \left[\frac{1}{jw} + \pi\delta(w)\right]$

$$= \frac{1}{2j}\left[\frac{1}{j(w-5)} - \frac{1}{j(w+5)}\right] + \frac{\pi}{2j}[\delta(w-5) - \delta(w+5)]$$

$$Y(jw) = H(jw)X(jw) = \frac{1}{2j}\left[\frac{1}{j(w-5)}\frac{1}{jw+4} - \frac{1}{j(w+5)}\frac{1}{jw+4}\right] + \frac{\pi}{2j}\left[\frac{1}{j5+4}\delta(w-5) - \frac{1}{-j5+4}\delta(w+5)\right]$$

3. $F^{-1}\{Y(jw)\} = F^{-1}\left\{\frac{1}{2j}\left[\frac{1}{j(w-5)}\frac{1}{jw+4} - \frac{1}{j(w+5)}\frac{1}{jw+4}\right] + \frac{\pi}{2j}\left[\frac{1}{j5+4}\delta(w-5) - \frac{1}{-j5+2}\delta(w+5)\right]\right\}$

$$= F^{-1}\left\{\frac{1}{2j}\left[\frac{(j5+4)^{-1}}{j(w-5)} + \frac{(-j5-4)^{-1}}{jw+4} - \frac{(-j5+4)^{-1}}{j(w+5)} - \frac{(j5-4)^{-1}}{jw+4}\right] + \frac{\pi}{2j(j5+4)}\delta(w-5) - \frac{\pi}{2j(-j5+4)}\delta(w+5)\right\}$$

$$= F^{-1}\left\{\frac{1}{2j(j5+4)}\left[\frac{1}{j(w-5)} + \pi\delta(w-5)\right] + \frac{1}{-2j(-j5+4)}\left[\frac{1}{j(w+5)} + \pi\delta(w+5)\right] + \frac{5/41}{jw+4}\right\}$$

$$= \left\{\frac{1}{2j(j5+4)}e^{j5t}u(t) + \frac{1}{-2j(-j5+4)}e^{-j5t}u(t) + \frac{5}{41}e^{-4t}u(t)\right\}$$

$$= 2\,\mathrm{Re}\left\{\frac{1}{2j(j5+4)}e^{j5t}u(t)\right\} + \frac{5}{41}e^{-4t}u(t)$$

$$= \mathrm{Re}\left\{\frac{1}{\sqrt{(16+25)}}e^{j5t + j\tan^{-1}(4/5) - 180}u(t)\right\} + \frac{5}{41}e^{-4t}u(t)$$

$$= \mathrm{Re}\left\{\frac{1}{\sqrt{41}}\cos(5t + \tan^{-1}(4/5) - 180^{o})u(t)\right\} + \frac{5}{41}e^{-4t}u(t)$$

4. $L\{x\} = \frac{5}{s^2+25}, ROC = \{\mathrm{Re}\ s > 0\};$

$$Y(s) = H(s)X(s) = \frac{1}{s+4}\frac{5}{s^2+25}, ROC = \{\mathrm{Re}\ s > 0\}$$

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{s+4}\frac{5}{s^2+25}\right\} = L^{-1}\left\{\frac{A}{s+4} + \frac{Bs+C}{s^2+25}\right\}$$

$Note: A = 5/[(-4)^2 + 25] = 5/41;\ B, C: As^2 + 25A + Bs^2 + 4Bs + Cs + 4C = 5 \Rightarrow B = -5/41, C = 4/41$

$$= L^{-1}\left\{\frac{5/41}{s+4}\right\} + L^{-1}\left\{\frac{-5/41s}{s^2+25} + \frac{4/41}{s^2+25}\right\} = \left\{\frac{5}{41}e^{-4t}u(t) - \frac{5}{41}\cos(5t)u(t) + \frac{4}{205}\sin(5t)u(t)\right\}$$

**Problem 2:**

Consider the continuous time causal filter with transfer function

$$H(s) = \frac{s-1}{(s+1)(s-2)}$$

1. Compute the response of the filter to $x[t] = u[t]$.

2. Compute the response of the filter to $x[t] = u[-t]$.

3. Repeat  parts 1 and 2 for a stable system with the same transfer function.

$$1.\, Causality \Rightarrow ROC_H = \{\operatorname{Re} s > 2\}$$

$$Y(s) = H(s)X(s) = \frac{s-1}{(s+1)(s-2)s};\ ROC \supseteq \{\operatorname{Re} s > 2\} \cap \{\operatorname{Re} s > 0\} = \{\operatorname{Re} s > 2\}$$

$$= \left\{\frac{1/2}{s}\right\}_{ROC=\operatorname{Re} s>0} + \left\{\frac{2/3}{(s+1)}\right\}_{ROC=\operatorname{Re} s>-1} + \left\{\frac{1/6}{(s-2)}\right\}_{ROC=\operatorname{Re} s>2}$$

$$y(t) = \frac{1}{2}u(t) + \frac{2}{3}e^{-t}u(t) + \frac{1}{6}e^{2t}u(t)$$

$$2.\, Y(s) = H(s)X(s) = \frac{s-1}{(s+2)(s-1)(-s)};\ ROC \supseteq \{\operatorname{Re} s > 2\} \cap \{\operatorname{Re} s < 0\} = \{\varnothing\}$$

$y(t)$ not well-defined.

$$3.\, Stability \Rightarrow ROC_H = \{-1 < \operatorname{Re} s < 2\}$$

$$3.1: Y(s) = H(s)X(s) = \frac{s-1}{(s+1)(s-2)s};\ ROC \supseteq \{-1 < \operatorname{Re} s < 2\} \cap \{\operatorname{Re} s > 0\} = \{0 < \operatorname{Re} s < 2\}$$

$$= \left\{\frac{1/2}{s}\right\}_{ROC=\operatorname{Re} s>0} + \left\{\frac{2/3}{(s+1)}\right\}_{ROC=\operatorname{Re} s>-1} + \left\{\frac{1/6}{(s-2)}\right\}_{ROC=\operatorname{Re} s<2}$$

$$y(t) = \frac{1}{2}u(t) + \frac{2}{3}e^{-t}u(t) + \frac{1}{6}[-e^{2t}u(-t)] = \frac{1}{2}u(t) + \frac{2}{3}e^{-t}u(t) - \frac{1}{6}e^{2t}u(-t)$$

$$3.2: Y(s) = H(s)X(s) = \frac{s-1}{(s+1)(s-2)(-s)};\ ROC \supseteq \{-1 < \operatorname{Re} s < 2\} \cap \{\operatorname{Re} s < 0\} = \{-1 < \operatorname{Re} s < 0\}$$

$$= -\left\{\frac{1/2}{s}\right\}_{ROC=\operatorname{Re} s<0} - \left\{\frac{2/3}{(s+1)}\right\}_{ROC=\operatorname{Re} s>-1} - \left\{\frac{1/6}{(s-2)}\right\}_{ROC=\operatorname{Re} s<2}$$

$$y(t) = -\frac{1}{2}[-u(-t)] - \frac{2}{3}e^{-t}u(t) - \frac{1}{6}[-e^{2t}u(-t)] = \frac{1}{2}u(-t) - \frac{2}{3}e^{-t}u(t) + \frac{1}{6}e^{2t}u(-t)$$

**Problem 3:**

Consider the discrete time stable filter with transfer function

$$H(z) = \frac{z-1}{(z-0.5)(z-2)}$$

1. Compute the response of the filter to $x[n] = u[n]$.

2. Repeat part 1 for a causal filter with the same transfer function.

$$1.\ Stability \Rightarrow ROC_H = \{0.5 <|z|< 2\}$$

$$Y(z) = H(z)X(z) = \frac{(z-1)z}{(z-0.5)(z-2)(z-1)};\ ROC \supseteq \{0.5 <|z|< 2\} \cap \{1 <|z|\}$$

$$= \left\{\frac{-1/3}{z-0.5}\right\}_{ROC=0.5<|z|} + \left\{\frac{4/3}{z-2}\right\}_{ROC=|z|<2} + \left\{\frac{0}{z-1}\right\}_{ROC=1<|z|}$$

$$y(n) = -\frac{1}{3}0.5^{n-1}u(n-1) - \frac{4}{3}2^{n-1}u(-n)$$

$$2.\ Causality \Rightarrow ROC_H = \{2 <|z|\}$$

$$Y(z) = H(z)X(z) = \frac{(z-1)z}{(z-0.5)(z-2)(z-1)};\ ROC \supseteq \{2 <|z|\} \cap \{1 <|z|\}$$

$$= \left\{\frac{-1/3}{z-0.5}\right\}_{ROC=0.5<|z|} + \left\{\frac{4/3}{z-2}\right\}_{ROC=2<|z|} + \left\{\frac{0}{z-1}\right\}_{ROC=1<|z|}$$

$$y(n) = -\frac{1}{3}0.5^{n-1}u(n-1) + \frac{4}{3}2^{n-1}u(n-1)$$

**EEE 304 Test 2 Name: ______SOLUTIONS____**

**Problem 1:**

For the continuous-time causal system with transfer function $H(s) = \frac{(-0.1s+1)}{(2s+1)}$ compute the following:

1. The amplitude and the phase of the steady-state response to a sinusoid input *x(t) = sin(10t+30$^o$)u(t).*
2. The discrete-time equivalent of H(s), say G(z), using the Backward Euler Approximation and a sampling interval of T = 0.1s.
3. For G(z), compute the amplitude and the phase of the steady-state response to the sinusoid x(t) sampled at the time instants nT, i.e., *x(n) = sin(n+30$^o$)u(n)*

1. The steady-state response is $y_{ss}(t) = |H(j10)|\sin(10t + 30° + \angle H(j10))$, so the amplitude is

$$ampl. = |H(j10)| = \sqrt{\frac{[(-0.1)(10)]^2 + 1}{[(2)(10)]^2 + 1}} = \sqrt{\frac{2}{401}} = 0.0706$$

$$\angle H(j10) = \tan^{-1}\left(\frac{(-0.1)(10)}{1}\right) - \tan^{-1}\left(\frac{(2)(10)}{1}\right) = -\tan^{-1}(1) - \tan^{-1}(20) = -45 - 87 = -132°$$

$$phase = 30 - 132 = -102°$$

2. Backward Euler uses the substitution *s = (z-1)/Tz*

$$G(z) = H(s)\Big|_{s=\frac{z-1}{Tz}} = \frac{\left(-0.1\frac{z-1}{0.1z}+1\right)}{\left(2\frac{z-1}{0.1z}+1\right)} = \frac{0.1}{(2.1z-2)}$$

3. The steady-state response is $y_{ss}(n) = |G(e^{j1})|\sin(n + 30° + \angle G(e^{j1}))$, so the amplitude is

$$ampl. = |G(e^{j1})| = \frac{|0.1|}{|(2.1e^{j1} - 2)|} = \frac{0.1}{\sqrt{(2.1\cos(1) - 2)^2 + (2.1\sin(1))^2}} = \frac{0.1}{\sqrt{3.872}} = 0.0508$$

$$\angle G(e^{j1}) = -\tan^{-1}\left(\frac{2.1\sin(1)}{2.1\cos(1) - 2}\right) = -\tan^{-1}\left(\frac{1.767}{-0.865}\right) = -\tan^{-1}\left(-\frac{1.767}{0.865}\right) - 180 =$$

$$= \tan^{-1}(2.042) - 180 = -116.1°$$

$$phase = 30° + \angle G(e^{j1}) = -86.1°$$

Notice that the sampling rate (10Hz) is not too fast relative to the sinusoid of frequency 10rad/s (1.6Hz) so that there is appreciable relative difference between continuous and discrete response. Their absolute difference is still small, however, since the sinusoid is well-above the system bandwidth (0.5rad/s).

**EEE 304** **Homework 3** **SOLUTIONS**

**Problem 1:**
Consider the following systems:

1. Transfer function $H(s) = \dfrac{s-1}{(s+2)(s+4)}$ (Continuous time, causal)

2. Transfer function $H(z) = \dfrac{z-1}{(z-0.5)(z-0.8)}$ (Discrete time, causal)

Compute the following:

1. Bode plot (expression, graph)
2. Response to sin(t) (for CT) and sin(2πn/10) (for DT)

$$1. |H(jw)| = \frac{\sqrt{w^2+1}}{\sqrt{w^2+4}\sqrt{w^2+16}},$$

$$\angle H(jw) = \tan^{-1}\left(\frac{\mathrm{Im}=w}{\mathrm{Re}=-1}\right) - \tan^{-1}\left(\frac{w}{2}\right) - \tan^{-1}\left(\frac{w}{4}\right) = 180 - \tan^{-1}\left(\frac{w}{1}\right) - \tan^{-1}\left(\frac{w}{1}\right) - \tan^{-1}\left(\frac{w}{2}\right)$$

$\Rightarrow$

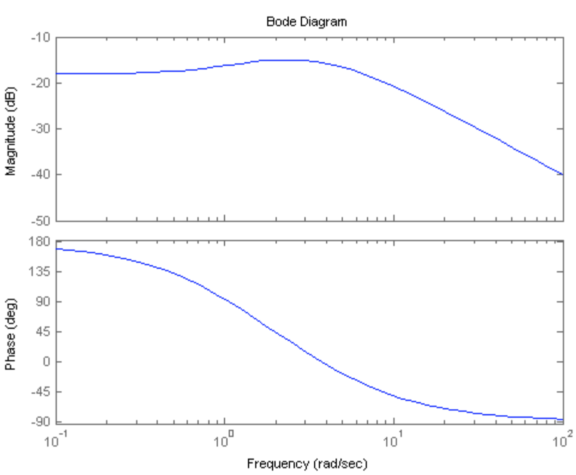
$$y(t) = |H(j1)|\sin(t + H(j1)) = 0.1534\sin(t + 94.4°)$$

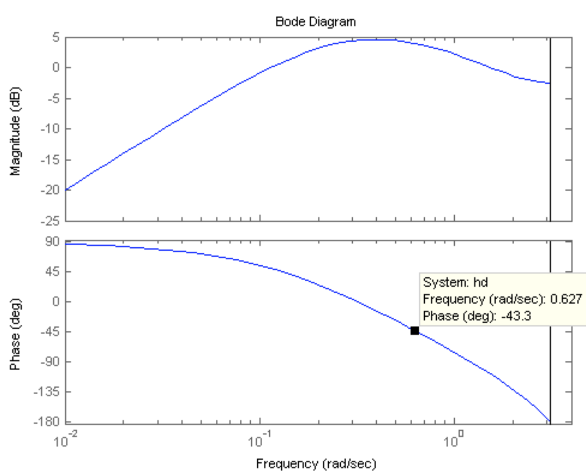
$$2. |H(e^{j\Omega})| = \frac{|e^{j\Omega}-1|}{|e^{j\Omega}-0.5||e^{j\Omega}-0.8|} = \frac{\sqrt{(\cos\Omega-1)^2+(\sin\Omega)^2}}{\sqrt{(\cos\Omega-0.5)^2+(\sin\Omega)^2}\sqrt{(\cos\Omega-0.8)^2+(\sin\Omega)^2}},$$

$$\angle H(e^{j\Omega}) = l180 + \tan^{-1}\left(\frac{\sin\Omega}{\cos\Omega-1}\right) - \tan^{-1}\left(\frac{\sin\Omega}{\cos\Omega-0.5}\right) - \tan^{-1}\left(\frac{\sin\Omega}{\cos\Omega-0.8}\right); \quad l = \#\,Negative\ \mathrm{Re}\ parts$$

$\Rightarrow$

$$y(n) = \left|H\left(e^{j\frac{\pi}{5}}\right)\right|\sin\left(\frac{\pi}{5}n + \angle H\left(e^{j\frac{\pi}{5}}\right)\right) = 1.583\sin\left(\frac{\pi}{5}n - 43.4°\right)$$



**Problem 2:**

1. Use forward and backward Euler approximations of derivative to derive the DT counterparts of the system with transfer function $H(s) = \dfrac{-0.2s+1}{(s+1)(s+2)}$, for sampling times 0.1, 1, 10.

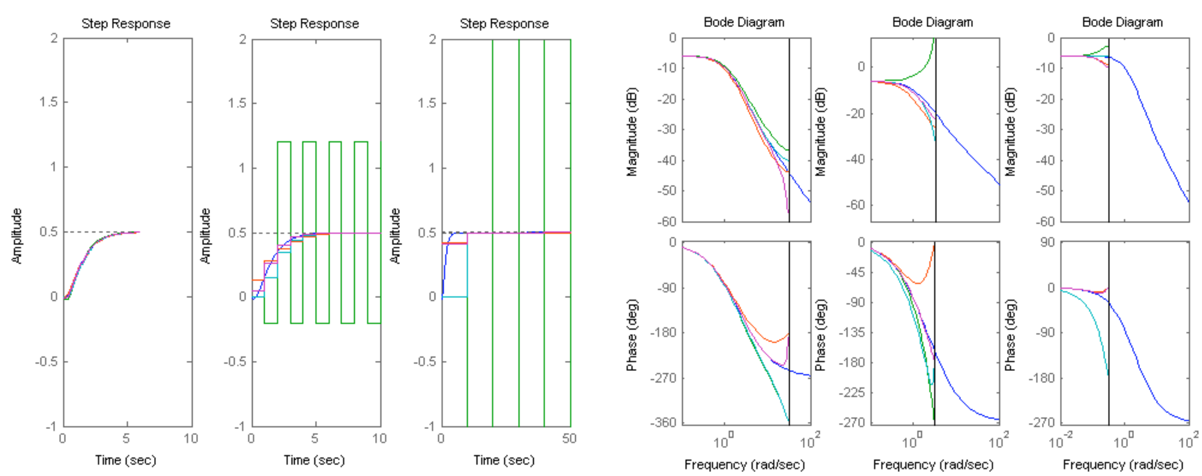
2. Use MATLAB to compare the step responses and frequency responses of the discretizations in P.2.1 with the CT transfer function, and its descretization using the function c2d, with zoh and foh options. Briefly, describe your observations.

$$Forward-Euler: s = \frac{z-1}{T} \Rightarrow H_d(z) = \frac{-0.2\dfrac{z-1}{T}+1}{\left(\dfrac{z-1}{T}+1\right)\left(\dfrac{z-1}{T}+2\right)} = \frac{(-0.2z+0.2+T)T}{(z-1+T)(z-1+2T)}$$

$$= \frac{-0.2Tz+(0.2+T)T}{(z-(1-T))(z-(1-2T))} = \begin{cases} \dfrac{-0.02z+0.03}{(z-0.9)(z-0.8)} & T=0.1 \\ \dfrac{-0.2z+1.2}{(z)(z+1)} & T=1 \\ \dfrac{-2z+102}{(z+9)(z+19)} & T=10 \end{cases}$$

$$Backward-Euler: s = \frac{1-z^{-1}}{T} \Rightarrow H_d(z) = \frac{-0.2\dfrac{z-1}{zT}+1}{\left(\dfrac{z-1}{Tz}+1\right)\left(\dfrac{z-1}{Tz}+2\right)} = \frac{((-0.2z+0.2+Tz)Tz}{(z-1+Tz)(z-1+2Tz)}$$

$$= \frac{((T-0.2)z+0.2)Tz}{((1+T)z-1)((1+2T)z-1)} = \begin{cases} \dfrac{(-0.01z+0.02)z}{(1.1z-1)(1.2z-1)} & T=0.1 \\ \dfrac{(0.8z+0.2)z}{(2z-1)(3z-1)} & T=1 \\ \dfrac{(98z+2)z}{(11z-1)(21z-1)} & T=10 \end{cases}$$



Observations: From the bode plots, the CT-DT frequency response approximation starts deteriorating about 1 decade before the half-sampling frequency (max Nyquist rate). The Forward Euler generates a frequency response approximation but the system is unstable. From the rest, the ZOH generates the max lag while the FOH is usually (not always) the best approximation. In the step responses, the ZOH shows as one full step behind (accurate at sampling times only) and the FOH zig-zags about the continuous response. The F-Euler instability is clear for T=1,10. (Zoom in the plots to see details).

MATLAB COMMAND HISTORY

```
T=.1;num=T*[T-0.2  0.2 0];den=conv([1+T -1],[1+2*T,-1]);Hdb=tf(num,den,T)
T=.1;num=[-0.2*T 0.2*T+T*T];den=conv([1 -1+T],[1 -1+2*T]);Hdf=tf(num,den,T)
T=.1;Hdzoh=c2d(H,T,'zoh'),Hdfoh=c2d(H,T,'foh')
subplot(131)
step(H,Hdf,Hdb,Hdzoh,Hdfoh)
axis([0,10,-1 2])
T=1;Hdzoh=c2d(H,T,'zoh'),Hdfoh=c2d(H,T,'foh')
T=1;num=[-0.2*T 0.2*T+T*T];den=conv([1 -1+T],[1 -1+2*T]);Hdf=tf(num,den,T)
T=1;num=T*[T-0.2  0.2 0];den=conv([1+T -1],[1+2*T,-1]);Hdb=tf(num,den,T)
subplot(132)
step(H,Hdf,Hdb,Hdzoh,Hdfoh)
axis([0,10,-1 2])
T=10;num=T*[T-0.2  0.2 0];den=conv([1+T -1],[1+2*T,-1]);Hdb=tf(num,den,T)
T=10;num=[-0.2*T 0.2*T+T*T];den=conv([1 -1+T],[1 -1+2*T]);Hdf=tf(num,den,T)
T=10;Hdzoh=c2d(H,T,'zoh'),Hdfoh=c2d(H,T,'foh')
subplot(133)
step(H,Hdf,Hdb,Hdzoh,Hdfoh)
axis([0,50,-1 2])

clf

T=.1;num=T*[T-0.2  0.2 0];den=conv([1+T -1],[1+2*T,-1]);Hdb=tf(num,den,T)
T=.1;num=[-0.2*T 0.2*T+T*T];den=conv([1 -1+T],[1 -1+2*T]);Hdf=tf(num,den,T)
T=.1;Hdzoh=c2d(H,T,'zoh'),Hdfoh=c2d(H,T,'foh')
subplot(131)
bode(H,Hdf,Hdb,Hdzoh,Hdfoh)
T=1;Hdzoh=c2d(H,T,'zoh'),Hdfoh=c2d(H,T,'foh')
T=1;num=[-0.2*T 0.2*T+T*T];den=conv([1 -1+T],[1 -1+2*T]);Hdf=tf(num,den,T)
T=1;num=T*[T-0.2  0.2 0];den=conv([1+T -1],[1+2*T,-1]);Hdb=tf(num,den,T)
subplot(132)
bode(H,Hdf,Hdb,Hdzoh,Hdfoh)
T=10;num=T*[T-0.2  0.2 0];den=conv([1+T -1],[1+2*T,-1]);Hdb=tf(num,den,T)
T=10;num=[-0.2*T 0.2*T+T*T];den=conv([1 -1+T],[1 -1+2*T]);Hdf=tf(num,den,T)
T=10;Hdzoh=c2d(H,T,'zoh'),Hdfoh=c2d(H,T,'foh')
subplot(133)
bode(H,Hdf,Hdb,Hdzoh,Hdfoh)
```

**EEE 304** **Test 3** **NAME:_______________**

**Problem 1:**
Suppose that a continuous time signal z(t) has Z(jw) that is bandlimited to $w_0$. The signal is filtered by a low-pass 2nd order Butterworth filter with cutoff frequency $w_0/2$. Let the output of the filter be x(t).


Find the sampling rate to allow perfect reconstruction of x(t) with an ideal low pass filter. (Explain.)

The Butterworth filter attenuates frequencies greater than $w_0/2$, but it does not reject them completely. In fact, using the high frequency asymptote for the magnitude $(w/w_c)^{-2}$, where the cutoff frequency is $w_0/2$, the magnitude at $w_0$ is approximately ¼. (A more exact computation yields 0.2425). That is the frequencies of z(t) between $w_0/2$ and $w_0$ are attenuated by at most a factor of 4 and cannot be considered as negligible. Hence, in general, X(jw) is only guaranteed to be bandlimited to $w_0$, (same as Z). Then, for perfect reconstruction, the sampling rate should be at least $2w_0$.

**Problem 2:**
Find the largest sampling interval $T_s$ to allow perfect reconstruction of the signals

1. $\sin 5t \cos 2t$

2. $\sin t - \sin 2t$

3. $te^{-t}u(t)$

1. The Fourier transform of the product signal is the convolution of the Fourier transforms of the individual signals, so the maximum frequency is 5+2 = 7. Then, the largest sampling interval is $T_s = \pi/7$.

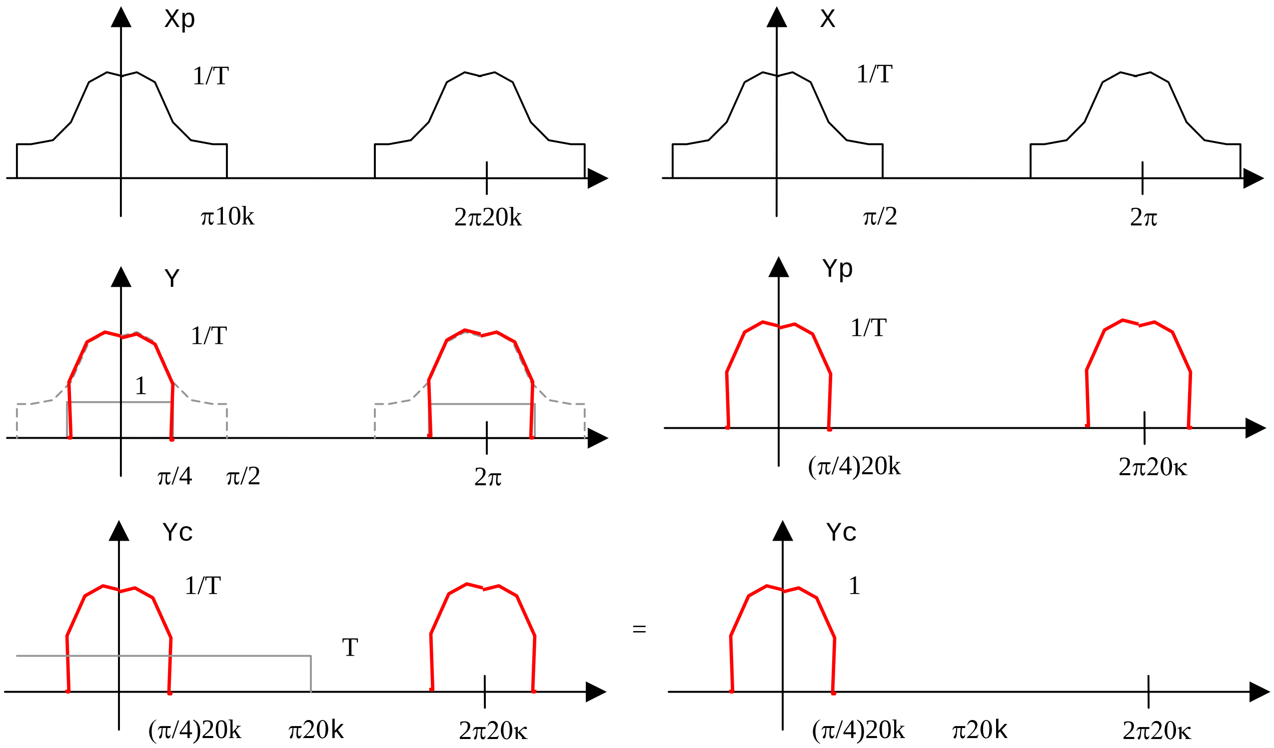
2. The Fourier transform of the summation signal is the summation of the Fourier transforms of the individual signals, which has maximum frequency max(1,2) = 2. Then, the sampling interval is $T_s = \pi/2$.

3. The Fourier transform of the signal can be found in the tables as $1/(jw+1)^2$. This is not a bandlimited signal so perfect reconstruction is not possible for any nonzero $T_s$ ($T_s \to 0$).

# EEE 304 HW 4

**Problem 1:**
Do Problems 7.29, 7.31 from the textbook.

**7.29**




**7.31**
Consider a test signal $x_c(t)=\exp(jwt)$, with $w < \pi/T$. Then, following the operations in Fig.P.7.31,

$x(n) = e^{jwnT} = \left(e^{jwT}\right)^n$

$y(n) = \frac{1}{2} y(n-1) + x(n) = H(z)\big|_{z=e^{jwT}} \left(e^{jwT}\right)^n$ (since x is an exponential)

$= \frac{1}{1 - \frac{1}{2} z^{-1}}\Bigg|_{z=e^{jwT}} \left(e^{jwT}\right)^n = \frac{2}{2 - e^{-jwT}} \left(e^{jwT}\right)^n$

$y_c(t) = Lowpass\left[\frac{2}{2 - e^{-jwT}} e^{jwnT}\right] = \frac{2}{2 - e^{-jwT}} e^{jwt}$ (because $w < \pi/T$)

$\Rightarrow H(jw) = \frac{2}{2 - e^{-jwT}}$; (since, for an LTI system with an exponential input $x_c(t) = e^{jwt}$, $y_c(t) = H(jw)e^{jwt}$)

Notice that from the last expression, the transfer function is H(s) = 2/(2-exp(-sT)) which is not a finite dimensional system.

**Problem 2:**

Find the largest sampling interval $T_s$ to allow perfect reconstruction of the signals (x*y denotes convolution)

1. $\frac{\sin t}{t} \sin t \Rightarrow w_{\max} = 1 + 1 \Rightarrow T_s = \frac{\pi}{w_{\max}} = \frac{\pi}{2}$ (using convolution of Fourier transforms).
2. $\frac{\sin 2t}{t} * \sin 3t \Rightarrow w_{\max} = 0 \Rightarrow T_s = \frac{\pi}{w_{\max}} \to \infty$ (using the filtering property, any Ts will do).
3. $\frac{\sin 2t}{t} \sin 3t \Rightarrow w_{\max} = 2 + 3 \Rightarrow T_s = \frac{\pi}{w_{\max}} = \frac{\pi}{5}$.
4. $\frac{\sin 4t}{t} * \sin 2t \Rightarrow w_{\max} = 2 \Rightarrow T_s = \frac{\pi}{w_{\max}} = \frac{\pi}{2}$.

**EEE 304** **TEST 4** **NAME:_______________**

**Problem:**

**a.** Determine the signal produced if the following sequence of operations is performed on a signal x(t) that is bandlimited to $w_m$ (i.e., X(jw) = 0 for |w|> $w_m$).

1. Modulation with a square wave carrier of frequency $3w_m$ and an unknown duty cycle, i.e.:

$$s(t) = \begin{cases} 1 & |t| < d \\ 0 & otherwise \end{cases} \quad where \quad d \in (0, T/2), \; and \; T \; is \; the \; period \; of \; s(t)$$

2. Bandpass filtering with an ideal filter H(jw) = 1 for $2w_m$ <|w|< $3w_m$.
3. Modulation with the same square wave carrier.
4. Lowdpass filtering with an ideal filter H(jw) = 1 for |w|< $w_m$.

**b.** How does the duty cycle parameter d affect the output signal?

**a.**

1. From Fourier tables (or Fourier transform via Fourier Series expansion)

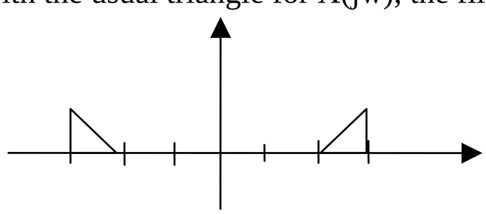
$$S(j\omega) = \sum_k \frac{2\sin kw_0T_1}{k}\delta(\omega - kw_0)\Big|_{w_0=3w_m,\,T_1=d} = \sum_k \frac{2\sin k3w_m d}{k}\delta(\omega - k3w_m)$$

$$X_s(j\omega) = \frac{1}{2\pi}X(j\omega) * S(j\omega) = \frac{1}{2\pi}\sum_k \frac{2\sin k3w_m d}{k} X\left(j(\omega - k3w_m)\right)$$

2. The filtering will allow only half of the first harmonic to pass. It will eliminate all other components and the DC.

$$X_H(j\omega) = H(j\omega)X_s(j\omega) = \begin{cases} \dfrac{\sin 3w_m d}{\pi}\left[X\left(j(\omega - 3w_m)\right) + X\left(j(\omega + 3w_m)\right)\right] & |\omega| < 3w_m \\ 0 & otherwise \end{cases}$$

For visualization, with the usual triangle for X(jw), the filtered signal $X_H$(jw) will be:



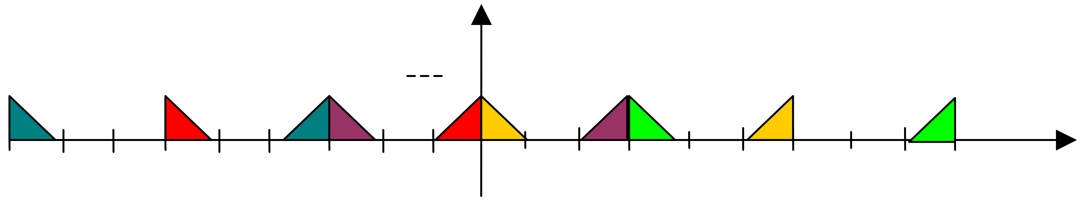
3. Modulating again, we get the same expression as $X_s$ but with $X_H$ in the place of X.

$$X_{Hs}(j\omega) = \frac{1}{2\pi}X_H(j\omega) * S(j\omega) = \frac{1}{2\pi}\sum_k \frac{2\sin k3w_m d}{k} X_H\left(j(\omega - k3w_m)\right)$$

$$= \sum_k \frac{\sin k3w_m d}{\pi k}\frac{\sin 3w_m d}{\pi}\left[X\left(j(\omega - k3w_m - 3w_m)\right) + X\left(j(\omega - k3w_m + 3w_m)\right)\right]$$

The low frequency signal is obtained for k = 1 and k = -1.

Visualization:



Color code: k = -2
k = -1
k = 0
k = 1
k = 2

4. After low-pass filtering the low frequency signal is

$$X_{LP}(j\omega) = \frac{\sin^2 3w_m d}{\pi^2} X(j\omega)$$

**b.** The output signal is

$$X_{LP}(j\omega) = \frac{\sin^2 3w_m d}{\pi^2} X(j\omega)$$

The duty cycle parameter d can take values in (0, T/2) while $3w_m$ is $2\pi/T$. So the sin argument ranges from 0 to $\pi$. At the two extremes the coefficient of X is zero since in one it is modulated by the zero signal and in the other by a constant. Both get filtered out by the bandpass filter. The peak is for d = T/4, for which the output amplitude is $1/\pi^2$ ~ 0.1 of the input (X).

**EEE 304 HW 5 SOLUTIONS**

**Problem 1:** Problem 8.24 from the textbook.

**(a)** With $\Delta = 0$,

$$S(j\omega) = \frac{2\pi}{T}\sum \delta\left(\omega - n\frac{2\pi}{T}\right), \quad X_s(j\omega) = \frac{1}{2\pi}S(j\omega) * X(j\omega) = \frac{1}{T}\sum X\left(j\omega - jn\frac{2\pi}{T}\right)$$

After filtering $X_s$ we get only the first side-lobes, multiplied with A:

$$Y(j\omega) = H(j\omega)X_s(j\omega) = \frac{A}{T}\left[X\left(j\omega - j\frac{2\pi}{T}\right) + X\left(j\omega + j\frac{2\pi}{T}\right)\right]$$

$$= \frac{2A}{T}\left\{\frac{1}{2}\left[X\left(j\omega - j\frac{2\pi}{T}\right) + X\left(j\omega + j\frac{2\pi}{T}\right)\right]\right\} = \frac{2A}{T}\mathbf{F}\left\{x(t)\cos\left(\frac{2\pi}{T}t\right)\right\}$$

**(b)** With $\Delta \neq 0$,

$$S(j\omega) = S^\circ(j\omega)e^{-j\omega\Delta} = \frac{2\pi}{T}\sum \delta\left(\omega - n\frac{2\pi}{T}\right)e^{-jn\frac{2\pi\Delta}{T}}$$

After filtering,

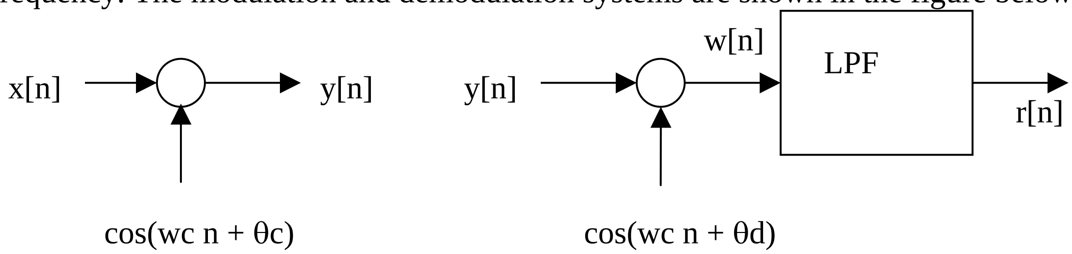
$$Y(j\omega) = H(j\omega)X_s(j\omega) = \frac{A}{T}\left[X\left(j\omega - j\frac{2\pi}{T}\right)e^{-jn\frac{2\pi\Delta}{T}} + X\left(j\omega + j\frac{2\pi}{T}\right)e^{jn\frac{2\pi\Delta}{T}}\right]$$

$$= \frac{2A}{T}\mathbf{F}\left\{x(t)\cos\left(\frac{2\pi}{T}t - \frac{2\pi\Delta}{T}\right)\right\}, \quad \Rightarrow \quad \omega_c = \frac{2\pi}{T}, \quad \theta_c = -\frac{2\pi\Delta}{T}$$

**(c)** The maximum allowable $\omega_M$ is determined by the "no-aliasing" condition so it is $\pi/T$.

Problem 8.47 and Solution

In this problem we want to consider the effect of a loss in synchronization in phase and/or frequency. The modulation and demodulation systems are shown in the figure below.




For parts (a) and (b) of this problem, the difference in frequency is zero, and the difference in the phase is denoted by $\Delta\theta = \theta d - \theta c$.

(a) If the spectrum is shown in Figure P8.47(b), sketch the spectrum of w[n]. (The spectrum is a symmetric triangle that is bandlimited at frequency wm.)
(b) Show that w can be chosen so that the output r[n] is $r[n] = x[n] \cos \Delta\theta$. In particular, what is r[n] if $\Delta\theta = \pi/2$?

Solution

Observe the following

$$y[n] = x[n] \cos (wc\, n + \theta c)$$
$$w[n] = y[n] \cos (wc\, n + \theta d)$$
$$w[n] = x[n] \cos (wc\, n + \theta c) \cos (wc\, n + \theta d)$$
$$w[n] = 0.5\, x[n] \cos \Delta\theta + 0.5\, x[n] \cos (2wc\, n + \theta c + \theta d)$$

Observe that in the frequency domain the first term in the equation for w[n] is a scaled version of the original signal and that the second term is a scaled and shifted version of the original signal. This observation can be used to sketch the result.

There are actually two cases here, depending on the magnitude of the phase shifts. One case has overlap the other does not.

Finally, if the carrier frequency is chosen correctly, it is possible to use an ideal low-pass filter (in the frequency domain) to filter out the second term in the equation for w[n]. In the special case where the phase shift is $\pi/2$, then the output r[n] is zero.

**Problem 3: (8.49 of textbook)**

**(a)** s(t) periodic so its Fourier transform is computed through the Fourier series expansion. That is,

$$s(t) = \sum a_k e^{jk\frac{2\pi}{T}t}, \quad \omega_o = \frac{2\pi}{T}$$

$$S(j\omega) = 2\pi \sum a_k \delta\left(\omega - k\frac{2\pi}{T}\right), \quad a_k = \frac{\sin k\omega_o T_1}{k\pi}, \quad T_1 = \frac{T}{4} \quad \Rightarrow a_k = \frac{\sin k\pi/2}{k\pi}$$

$$a_0 = 1/2$$

(The $a_k$ is zero for even k, other than 0.)

The modulated (chopped) x(t) has Fourier transform

$$X_s(j\omega) = \frac{1}{2\pi} X(j\omega) * S(j\omega) = \sum a_k X\left(j\omega - jk\frac{2\pi}{T}\right)$$

yielding the band-pass filtered-chopped signal, say v(t)

$$V(j\omega) = H_1(j\omega) \sum a_k X\left(j\omega - jk\frac{2\pi}{T}\right) = Aa_1 X\left(j\omega - j\frac{2\pi}{T}\right) + Aa_{-1} X\left(j\omega + j\frac{2\pi}{T}\right)$$

$$= \frac{A}{\pi} X\left(j\omega - j\frac{2\pi}{T}\right) + \frac{A}{\pi} X\left(j\omega + j\frac{2\pi}{T}\right)$$

The maximum allowable frequency content in x(t) for this expression to be valid is $\omega_M < \pi/T$.

Next, V(jω) is re-modulated (chopped) and low-pass filtered. The modulation by s(t) produces two replicas of X(jω) at 0, each multiplied by $1/\pi$, yielding a total coefficient of $2A/\pi^2$. The rest of the replicas are at $k2\pi/T$ that are filtered out by the $H_2$ low-pass filter. So,

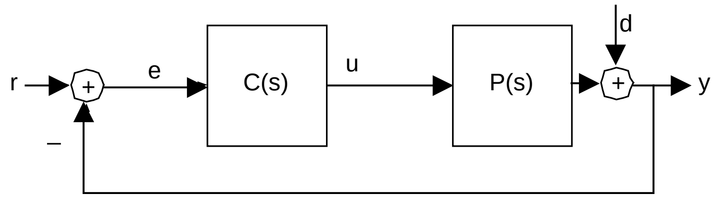
$$Y(j\omega) = H_2(j\omega) V_s(j\omega) = \frac{2A}{\pi^2} X(j\omega)$$

with the same condition on the maximum frequency in x(t), $\omega_M < \pi/T$.

**(b)** From the last expression, the equivalent gain of the overall system is $2A/\pi^2$.

**EEE 304** **Test 5** **Name: ___SOLUTIONS____**

**Problem 1:**
For the feedback system shown below, compute the transfer functions y/d, u/d.




$$\frac{y(s)}{d(s)} = \frac{1}{1+P(s)C(s)}, \quad \frac{u(s)}{d(s)} = \frac{-C(s)}{1+P(s)C(s)}$$

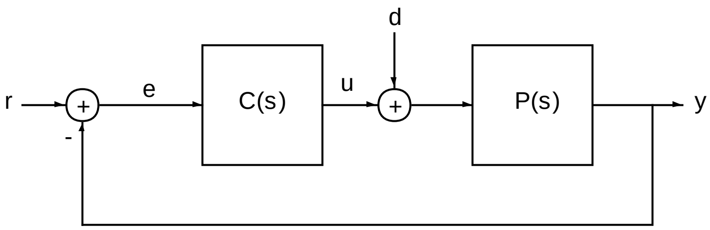
**Problem 2:**
For the feedback system of Problem 1, suppose $P(s)=1$ and $C(s)=K/s$.
1. Determine K so that the crossover frequency is 1.
2. For the same K, find the Phase Margin of the feedback system.

1. $\left| P(s)C(s)\big|_{s=jw_c, w_c=1} \right| = 1 \Rightarrow K = 1$

2. $PM = 180 + \angle PC(jw_c) = 180 - \tan^{-1}(1/0) = 90°$

**EEE 304** **HW 6** **SOLUTIONS**

**Problem 1:**
For the feedback system shown below, compute the transfer functions y/r, y/d, u/r, u/d.




$$\frac{y(s)}{r(s)}=\frac{PC}{1+PC},\quad \frac{y(s)}{d(s)}=\frac{P}{1+PC},\quad \frac{u(s)}{r(s)}=\frac{C}{1+CP},\quad \frac{u(s)}{d(s)}=\frac{-CP}{1+CP},$$

**Problem 2:**
For the feedback system of Problem 1, suppose $P(s)=1/(s+1)$.

a. When $C(s) = K$, design K so that the loop crossover frequency (i.e., w: $|P(jw)C(jw)| = 1$) is 10. What is the contribution of a constant unit disturbance to the output?
b. When $C(s)=K(Ts+1)/s$, design K,T so that the crossover frequency is 10 and the phase margin (i.e., the difference between the loop angle and –180 at the crossover frequency, $\angle P(jw_c)C(jw_c)+180$) is $50^o$. What is the contribution of a constant unit disturbance to the output?

a. $\omega_c=10: |P(j\omega_c)C(j\omega_c)|=1 \Rightarrow \left|\frac{K}{\sqrt{1+(\omega_c)^2}}\right|=1\Rightarrow |K|=\sqrt{101}=10.05$

$K>0$ for stability $(\angle C+\angle P>-180)$

$$y_d(s)=\frac{P}{1+PC}d(s)=\frac{1}{s+K+1}\frac{1}{s}=\frac{-1/11.05}{s+K+1}+\frac{1/11.05}{s}\Rightarrow y_d(t)=0.09\left(1-e^{-11.5t}\right)U(t)$$

$\Rightarrow y_{d,ss}=\lim_{t\to\infty} y_d(t)=0.09$

b. $\omega_c=10: \angle C+\angle P=-180+50=-130=\tan^{-1}(T\omega_c)-90-\tan^{-1}(\omega_c)$

$\Rightarrow \tan^{-1}(T\omega_c)=-44.3\Rightarrow T\omega_c=0.98\Rightarrow T=0.098$

$\omega_c=10: |P(j\omega_c)C(j\omega_c)|=1\Rightarrow \left|\frac{K\sqrt{1+(T\omega_c)^2}}{\omega_c\sqrt{1+(\omega_c)^2}}\right|=1\Rightarrow |K|=72$

$K>0$ for stability $(\angle C+\angle P>-180)$

$$y_d(s)=\frac{P}{1+PC}d(s)=\frac{s}{s^2+s+K(Ts+1)}\frac{1}{s}=\frac{1}{s^2+(1+KT)s+K}=\frac{1}{s^2+8.02s+72}$$

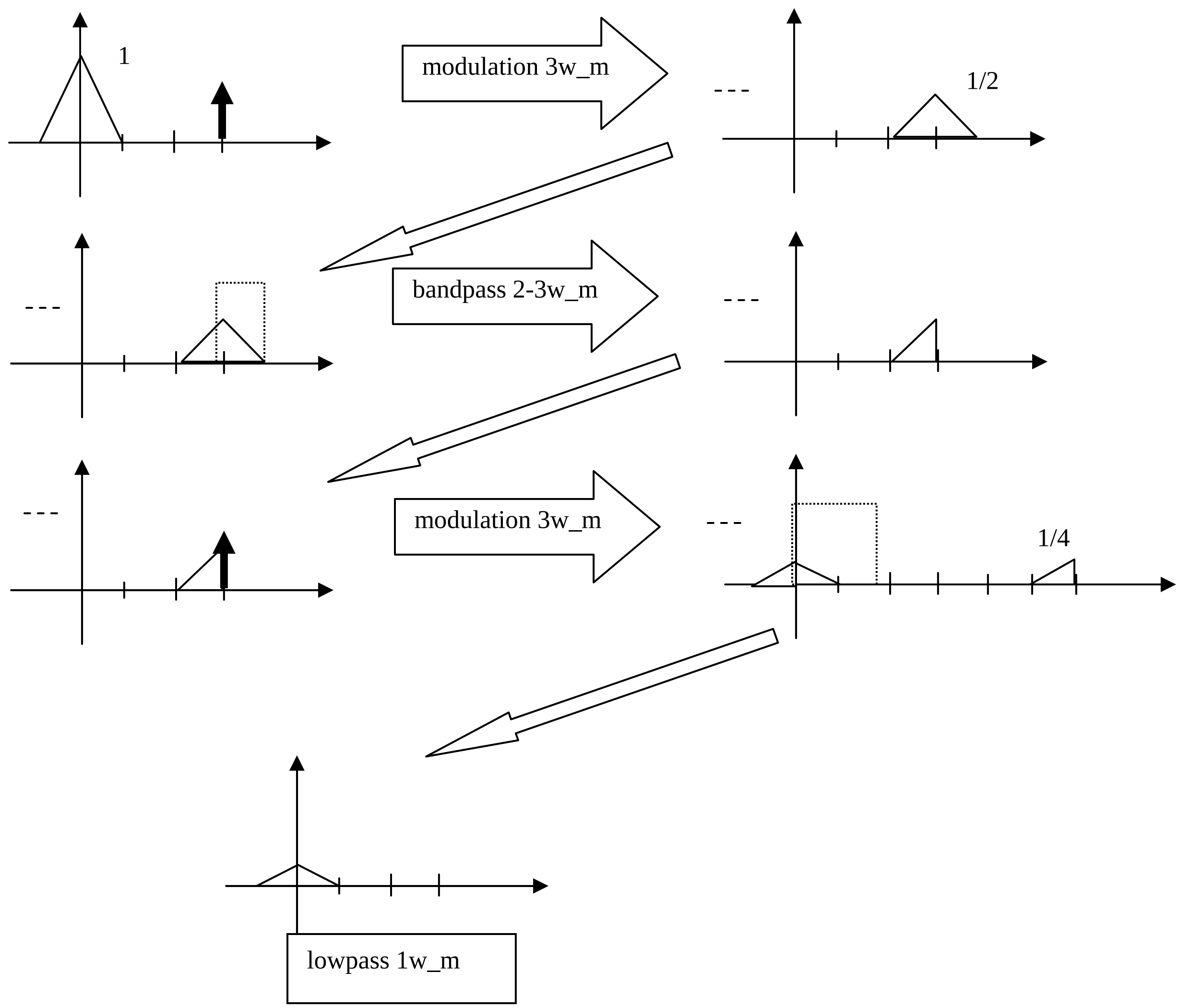
$$\overbrace{\Rightarrow}^{a=4,wo=7.5}\quad y_d(t)=0.13e^{-4t}\sin(7.5t)U(t)\Rightarrow y_{d,ss}=\lim_{t\to\infty}y_d(t)=0$$

**EEE 304** **TEST 4 SOLUTIONS**

**Problem 1:**

Determine the signal produced if the following sequence of operations is performed on a signal x(t) that is bandlimited to $w_m$ (i.e., X(jw) = 0 for |w|> $w_m$).

1. Modulation with a cosine carrier of frequency $3w_m$.
2. Bandpass filtering with an ideal filter H(jw) = 1 for $2w_m$ <|w|< 3 $w_m$.
3. Modulation with a cosine carrier of frequency $3w_m$.
4. Lowdpass filtering with an ideal filter H(jw) = 1 for |w|< $w_m$.




FINAL RESULT: ¼ x(t)

**EEE 304 Test 2 Name: ____SOLUTIONS______**

**Problem 1:**

Consider the following system with transfer function $H(s) = \dfrac{1}{(0.1s+1)^2}$ (Continuous time, causal)

1. Find the amplitude of the steady-state response to a sinusoid input $x(t) = \sin(5t+30^o)u(t)$.
2. Compute the discrete-time equivalent of H(s), say G(z), using the Backward Euler Approximation and a sampling interval of T = 0.01s.
3. For G(z), compute the amplitude of the steady-state response to the sinusoid x(t) sampled at the time instants nT, i.e., $x(n) = \sin(0.05n+30^o)u(n)$

1. The steady-state response is $y_{ss}(t) = |H(j5)|\sin(5t + 30° + \angle H(j5))$, so the amplitude is

$$|H(j5)| = \frac{1}{[(0.1)(5)]^2+1} = \frac{1}{1.25} = 0.8$$

2. Backward Euler uses the substitution s = (z-1)/Tz

$$G(z) = H(s)\Big|_{s=\frac{z-1}{Tz}} = \frac{1}{\left(0.1\dfrac{z-1}{0.01z}+1\right)^2} = \frac{0.01z^2}{(z-1+0.1z)^2} = \frac{0.01z^2}{(1.1z-1)^2}$$

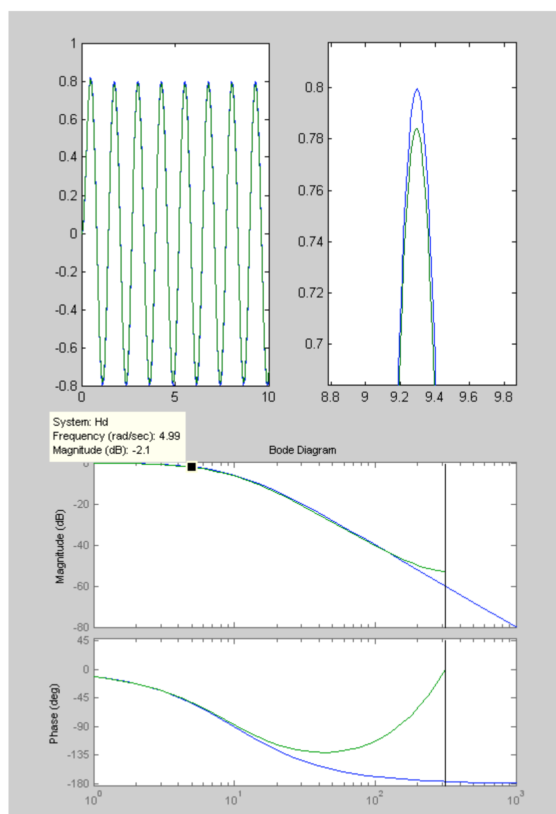
3. The steady-state response is $y_{ss}(n) = |G(e^{j0.05})|\sin(0.05n + 30° + \angle G(e^{j0.05}))$, so the amplitude is

$$|G(e^{j0.05})| = \frac{|0.01(e^{j0.05})^2|}{|(1.1e^{j0.05}-1)^2|} = \frac{|0.01(e^{j0.05})^2|}{|(1.1e^{j0.05}-1)^2|} = \frac{0.01}{(1.1\cos(0.05)-1)^2+(1.1\sin(0.05))^2} = \frac{0.01}{0.0127}$$
$$= 0.784$$

Notice that the sampling rate (100Hz) is fast relative to the filter bandwidth (10rad/s ~ 1.6Hz) so the discretization is fairly accurate. Also, the sampling of the sinusoid of frequency 5 rad/s at 100Hz is much faster than the Nyquist rate, indicating good agreement between continuous and discrete response.
You can view the results in MATLAB by issuing the following commands:

```
>> H=tf(1,[.1 1]);H=H*H
>> Hd=tf([.1 0],[1.1 -1],.01);Hd=Hd*Hd
>> t=[0:.01:10];
>> x=sin(5*t);
>> y=lsim(H,x,t);
>> plot(t,y)
>> yd=lsim(Hd,x,t);
>> plot(t,y,t,yd)
>> bode(H,Hd)
```

Also, keep in mind that MATLAB displays the Bode plots in terms of the continuous time frequency $\omega = \Omega / T$



**EEE 304** **Test 1** **NAME: ___________________**

**Problem 1:**
Compute the unit step response (x(t)=u(t)) of the system

$$H(s) = \frac{1}{(s-0.5)(s+2)}$$ (Continuous time, stable)

$$Y(s) = \frac{1}{(s-0.5)(s+2)s} \quad ROC = \{-2 < \operatorname{Re} s < 0.5\} \cap \{0 < \operatorname{Re} s\}$$

$$= \underbrace{\frac{-1}{s}}_{causal} + \underbrace{\frac{4/5}{(s-0.5)}}_{anticausal} + \underbrace{\frac{1/5}{(s+2)}}_{causal}$$

$$y(t) = -u(t) - \frac{4}{5}e^{0.5t}u(-t) + \frac{1}{5}e^{-2t}u(t)$$

**Problem 2:**
Compute the unit step response (x(n)=u(n)) of the system

$$H(z) = \frac{1}{(z-0.5)(z+2)}$$ (Discrete time, stable)

$$Y(z) = \frac{z}{(z-0.5)(z+2)(z-1)} \quad ROC = \{0.5 <| z |< 2\} \cap \{1 <| z |\}$$

$$= \underbrace{\frac{2/3}{z-1}}_{causal} + \underbrace{\frac{-4/15}{(z+2)}}_{anticausal} + \underbrace{\frac{-2/5}{(z-0.5)}}_{causal}$$

$$y(n) = \frac{2}{3}u(n-1) + \frac{4}{15}(-2)^{n-1}u(-n) - \frac{2}{5}0.5^{n-1}u(n-1)$$

**EEE 304** **Test 3** **NAME:________________**

**Problem 1:**
Suppose that a continuous time signal x(t) has X(jw) that is bandlimited to $w_0$. Find the sampling rate to allow perfect reconstruction with ideal low pass filters of dx/dt. (Explain.)

y = dx/dt is an LTI system, so it is described by a multiplication in the frequency domain. Since X(jw) is zero beyond w0, so is Y(jw). Then, for perfect reconstruction, the sampling rate should be at least $2w_0$.

**Problem 2:**
Find the largest sampling interval $T_s$ to allow perfect reconstruction of the signals (x*y denotes convolution)
1. $\sin t \cos 2t$ .
2. $\cos t * \sin 2t$

1. The Fourier transform of the composite signal is the convolution of the Fourier transforms of the individual signals, so the maximum frequency is 2+1 = 3. Then, the largest sampling interval is $T_s = \pi/3$.

2. The Fourier transform of the composite signal is the multiplication of the Fourier transforms of the individual signals, which is 0. Then, the sampling interval is $T_s$ can be arbitrarily large.