

EEE 304, HW2 SOLUTIONS

Problem 1:

Consider the following systems:

1. Transfer function $H(s) = \frac{s-10}{(s+1)(s+10)}$ (Continuous time, causal)
2. Transfer function $H(z) = \frac{0.1z-0.2}{z^2-0.9z}$ (Discrete time, causal)

Compute the following:

1. Bode plot (expression, graph)
2. Response to $\sin(2\pi t)$ (for CT) and $\sin(2\pi n/10)$ (for DT)

1.1. $|H(j\omega)| = \frac{\sqrt{\omega^2+10^2}}{\sqrt{\omega^2+1^2}\sqrt{\omega^2+10^2}}; \angle H(j\omega) = \text{atan}\left(-\frac{\omega}{10}\right) + \pi - \text{atan}\frac{\omega}{1} - \text{atan}\frac{\omega}{10}$

1.2. $|H(e^{j\Omega})| = \frac{\sqrt{(0.1\cos\Omega-0.2)^2+(0.1\sin\Omega)^2}}{(1)\sqrt{(\cos\Omega-0.9)^2+(\sin\Omega)^2}}; \angle H(j\omega) = \text{atan}\left(\frac{0.1\sin\Omega}{0.1\cos\Omega-0.2}\right) - \Omega - \text{atan}\left(\frac{\sin\Omega}{\cos\Omega-0.9}\right)$

The angle should be corrected by π for every atan denominator (real part) which is negative.

2.1 Evaluating the magnitude and angle for the frequency $\omega = 2\pi$ we get

$$y(t) = |H(j2\pi)| \sin(2\pi t + \angle H(j2\pi)) = 0.157 \sin(2\pi t + 34.8^\circ)$$

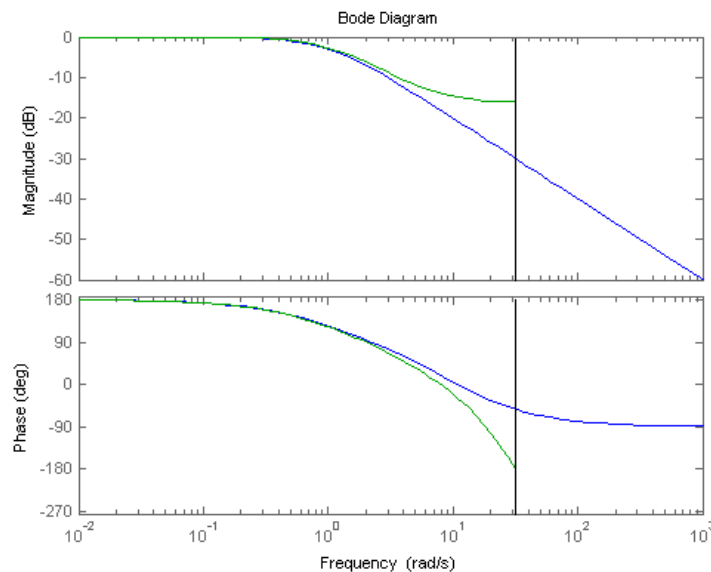
Note: We can also get the values with the MATLAB command

`>> [m,p] = bode(H,2*pi)`

2.2 Evaluating the magnitude and angle for the frequency $\omega = 2\pi/10$ we get

$$y(t) = |H(e^{j2\pi/10})| \sin\left(\frac{2\pi n t}{10} + \angle H(e^{j2\pi/10})\right) = 0.223 \sin\left(\frac{2\pi n t}{10} + 18.9^\circ\right)$$

The Bode plots are shown for the continuous frequency. For the DT-system, we take $T = 0.1$ and the discrete frequency is $\Omega = \omega T = 0.1\omega$



NOTE:

- The DT system is a discrete approximation of the continuous one with sampling time $T = 0.1$. The approximation is good up to frequencies ~ 0.1 (Nyquist).

Problem 2:

1. Use forward and backward Euler and Tustin approximations of derivative to derive the DT counterparts of the system of Problem 1.1, for sampling times 0.01, 0.1, 1.

2. Use MATLAB to compare the step responses and frequency responses of the discretizations in P.2.1 with the CT transfer function, and its discretization using the function c2d, with Tustin, zoh and foh options (sample code for a different problem is given below). Briefly, describe your observations.

We use the definitions for the Euler and Tustin transformations to write the expressions for the DT-equivalent transfer functions.

$$H_{fe}(z) = H(s)|_{s=\frac{z-1}{T}} = \frac{T(z-1-10T)}{(z-1+T)(z-1+10T)}$$

$$H_{be}(z) = H(s)|_{s=\frac{z-1}{zT}} = \frac{Tz(z-1-10Tz)}{(z-1+Tz)(z-1+10Tz)}$$

$$H_t(z) = H(s)|_{s=\frac{2(z-1)}{T(z+1)}} = \frac{T(z+1)(2(z-1)-10T(z+1))}{(2(z-1)+T(z+1))(2(z-1)+10T(z+1))}$$

A “trick” to compute the transfer functions is to define the variable ‘z’ as the discrete transfer function and then write the required transfer functions according to the derived formulae with the bilinear transformations. Thus, we evaluate the transfer functions with the following MATLAB commands:

```
T=1;z=tf([1 0],1,T)
Hf=T*(z-1-10*T)/(z-1+T)/(z-1+10*T)
Hb=T*z*(z-1-10*T*z)/(z-1+T*z)/(z-1+10*T*z)
Ht=T*(z+1)*(2*(z-1)-10*T*(z+1))/(2*(z-1)+T*(z+1))/(2*(z-1)+10*T*(z+1))
```

We find:

For $T = 1$

$$\begin{array}{l} H_f = \\ z - 11 \\ \hline z^2 + 9z \end{array}$$

$$\begin{array}{l} H_b = \\ -9z^2 - z \\ \hline 22z^2 - 13z + 1 \end{array}$$

$$\begin{array}{l} H_t = \\ -8z^2 - 20z - 12 \\ \hline 36z^2 + 12z - 8 \end{array}$$

For T=0.1

$$H_f = \frac{0.1z - 0.2}{z^2 - 0.9z}$$

$$H_b = \frac{-0.1z}{2.2z^2 - 3.1z + 1}$$

$$H_t = \frac{0.1z^2 - 0.2z - 0.3}{6.3z^2 - 7.8z + 1.9}$$

For T=0.01

$$H_f = \frac{0.01z - 0.011}{z^2 - 1.89z + 0.891}$$

$$H_b = \frac{0.009z^2 - 0.01z}{1.111z^2 - 2.11z + 1}$$

$$H_t = \frac{0.019z^2 - 0.002z - 0.021}{4.221z^2 - 7.998z + 3.781}$$

2. We evaluate the rest of the DT-equivalents using the command 'c2d':

`Hzoh=c2d(H,T,'zoh')`

`Hfoh=c2d(H,T,'foh')`

`Ht2=c2d(H,T,'tustin')` %% same as the previously derived

And use the 'bode' and 'step' commands to plot the frequency and step responses.

`bode(H,Hf,Hb,Ht,Hzoh,Hfoh)`

`step(H,Hf,Hb,Ht,Hzoh,Hfoh)`

We observe that the approximation of the continuous time responses improves as the sampling time decreases. In the frequency domain, the approximation is typically good up to one-tenth of Nyquist frequency; beyond that, Tustin usually provides the better approximations. For the step responses, the approximation of the continuous response also improves as the sampling time decreases. Only the Forward Euler has stability problems and can produce an unstable approximation if the sampling time is larger than twice the smallest time constant (here 0.1)

