EEE 481 Homework Problems

P.3.1

Consider the system $x(k+1) = 0.95x(k) + 0.05u(k)$, where the multiplications are quantized to 0.01. Use simulation to assess the mean, and variance of the error due to quantization (compared to non-quantized operations). Apply various inputs u(k), e.g., random, sinusoid, quantized to 0.01.

The multiplication quantization is modeled as random noise n(k) of uniform distribution ½ LSB. We will ignore addition. The system now is $x(k+1) = 0.95x(k) + 0.05u(k) + n_1(k) + n_2(k)$ .Then, the output contribution of that noise (x_n(k)) is described by the transfer function G(z)=1/(z-0.95) (the quantization occurs after taking the product 0.05u(k)) and is bounded as follows:

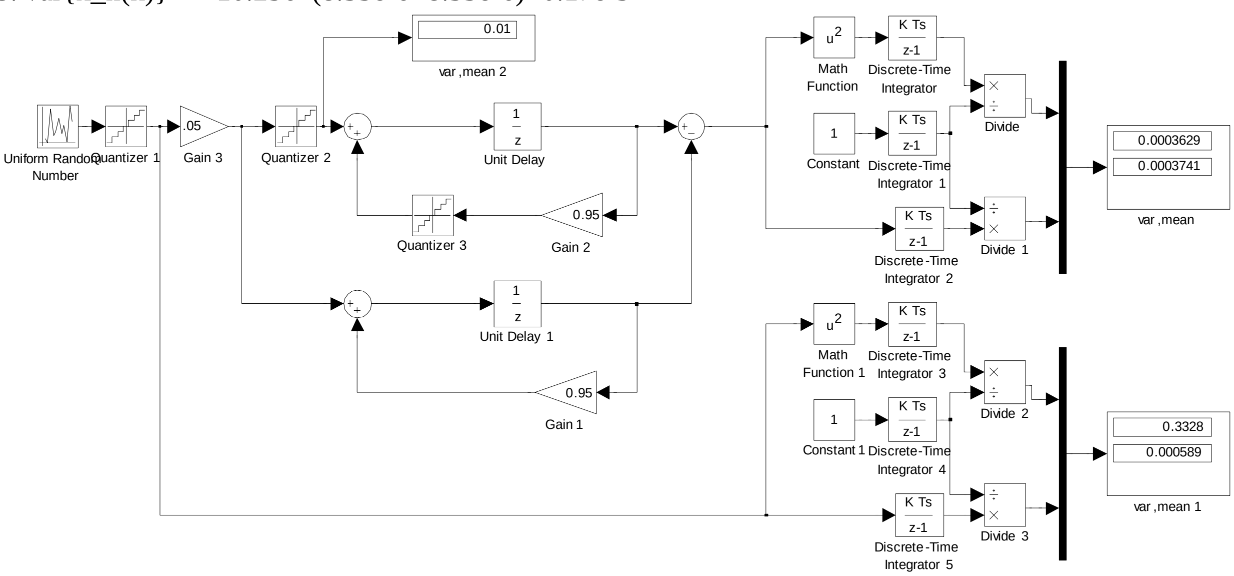
1. mean(x_n) = G(1)mean(n1)+ G(1)mean(n2)
2. var{x_n(k)} <= $|G(e^{j\Omega})|_2^2$var{n1(k)} +$|G(e^{j\Omega})|_2^2$var{n2(k)}

Notice that both quantization noises enter at the same node so there is only one transfer function.

Computing the theoretical estimates, G(1)=1/0.05=20. $|G|_2^2$ = sum$|g(k)|^2$=1/(1-0.95^2)=10.256. (The last one is based on Parseval, or simply using Matlab.)

For a round-off quantization, whose mean is 0 LSB, max(|n|)= ½ LSB = 0.005 and var = $(½ \text{ LSB})^2/3$ =8.33e-6. Thus,

1. mean(x_n) = 20*(0+0)=0
3. var{x_n(k)} <= 10.256*(8.33e-6+8.33e-6)=0.17e-3



The Simulink implementation of a simulator is shown above. We try different input signals: random numbers (uniform or gaussian), sinusoids, etc. In general, we must try different frequencies and different amplitudes as well, since the quantization makes the system nonlinear and the response to a scaled input is not simply the scaled response. The main results are tabulated below. Notice that, for stochastic inputs, the variance estimate is the least conservative one, but the estimate does not bound the actual signal, especially for low amplitude excitation where the nonlinearity is more prevalent (for input amplitude 0.05, the input to the quantized system is actually zero!). For slow sinusoids of small amplitude, the stochastic variance bound is too optimistic. This is where the more conservative norm(G,inf) estimate of the system gain becomes more appropriate. Also, it is possible that the mean is nonzero. In this case, the variance estimate should be adjusted (for simplicity, we skip the adjustment, keeping track of the mean to correctly interpret the results).

A priori estimate	Computed rand{-0.05,0.05}	Computed rand{-10,10}	Computed 0.05sin(0.001k)	Computed 10sin(0.001k)
var = 0.17e-3	0.022e-3	0.168e-3	1.28e-3	2.3e-3
mean = 0	-0.028e-3	0.32e-3	-0.07e-3	0.06e-3

As a last remark, the mean estimate becomes relevant if we use a different quantization scheme (floor, ceil) that have a nonzero mean. This estimate is fairly accurate.

P.3.2

Ziegler-Nichols Tuning: Apply the two Z-N methods from the notes to tune a PID for the plant $P(s)=\dfrac{5(-0.2s+1)}{s^2+3s+1}$. Compare the results with a PID designed for a gain crossover frequency of 2 rad/s and 50deg. phase margin.

*Hint: Define P as a transfer function object and use step(P) to get an estimate of R,L for the first Z-N tuning. Then iterate k on step(feedback(k*P,1)) until the system is marginally stable (slowly increasing or slowly decreasing response). Then estimate Ku,Pu for the second Z-N tuning. Define the compensators and compare step responses and bode plots for the transfer functions command-to-output and input disturbance-to-output*

```
P=tf(5*[-.2 1],[1 3 1])
step(P),grid
R=(2.4-.52)/(2.33-.8),L=0.37, %from graph, R=1.22
s=tf([1 0],1);
Kp=1.2/R/L;Ki=0.6/R/L/L;Kd=0.6/R;ZN1=Kp+Ki/s+Kd*s/(.01*s+1)
%use a fast pole for the pseudo-differentiator
step(fbk(P*5,1))
step(fbk(P*3,1))
step(fbk(P*2,1))
Ku=3,Pu=2.58-0.942,   %Pu=1.56
Kp=0.6*Ku;Ki=1.2*Ku/Pu;Kd=0.075*Ku*Pu;ZN2=Kp+Ki/s+Kd*s/(.01*s+1)

% design a pid using crossover/pm methods for a similar BW
[m,p]=bode(P*1/s/(.01*s +1),2)
Ph=-(p-360)-130
Tz=tan(Ph/2*pi/180)/2
C=tf(conv([Tz 1],[Tz 1]),[.01 1 0])
k=1/bode(P*C,2)
C=tf(conv([Tz 1],[Tz 1]),[.01 1 0])*k
%k=1.04, Tz=0.59,Ph=99.5
step(fbk(P*C,1),fbk(P*ZN1,1),fbk(P*ZN2,1))
bodemag(fbk(P*C,1),fbk(P*ZN1,1),fbk(P*ZN2,1))
bodemag(fbk(1,P*C),fbk(1,P*ZN1),fbk(1,P*ZN2))
```

Notice that both ZN methods yield much smaller phase margins than the classical design (20-30 deg). They do, however, offer smaller sensitivity at low frequencies without increasing the loop bandwidth too much. (They do increase the Sensitivity peak and resonance effect). The closed-loop test ZN is somewhat more reliable.

EEE 481 HW 3

P.1

The read arm on a computer disk drive has transfer function $H(s) = \dfrac{1000}{s^2}$.

1. Design an analog PID controller to achieve a bandwidth of approx. 100Hz with 50deg phase margin.
2. Design a digital PID with a sampling rate 1kHz and simulate the closed loop step response.
3. Keeping the same coefficients of the digital PID, perform a simulation study to determine approximate high/low limits of the sampling rate for which the closed loop is stable.

Solution:

1. This plant has constant phase –180deg. Taking the pseudo-derivative pole at 10-times the crossover frequency, corresponding to a phase delay of 5.7deg, the required phase lead from each zero is 145.7/2 deg. whose tangent is 3.24, so τ_z =3.24/w_c. For the crossover frequency we can pick the desired closed-loop BW (100Hz) as a first approximation. A better guess is w_c = BW/1.5 = 419 rad/s. For this selection, τ_p = 1/4190 = 2.4e-4 and τ_z = 7.7e-3. We substitute these values in the PID transfer function and evaluate the required gain for a crossover at 419: K=1/bode(H*C,419), which produces K=6.48e3 and a final compensator

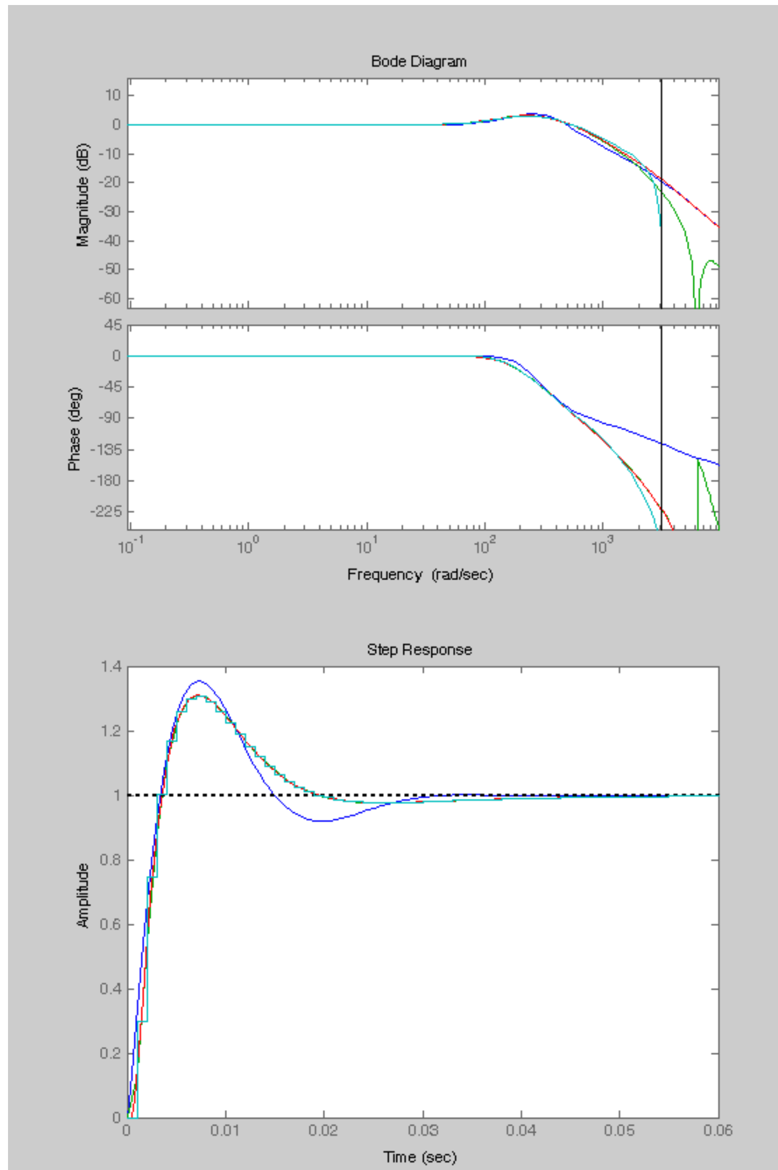
$$C(s) = \frac{0.3842\,s^2 + 99.79\,s + 6480}{0.00024\,s^2 + s}$$

2. The sampling time is Ts = 1/1000 s = 1e-3. Since the control signal is reconstructed with a ZOH, we add phase lag ($-w_cT_s/2$) at crossover = -12 deg. So the required phase lead is now 157.7 deg from the two zeros. Each zero should contribute τ_z =tan(157.7/2) /w_c = 1.2e-2. Notice the higher lead required due to the lag from the "slow" sampling rate). Now K=1/bode(Hd,Cd) = 2.8e3, where Hd is the plant with Ts/2 delay. So,

$$Cd(s) = \frac{0.41\,s^2 + 67.5\,s + 2813}{0.00024\,s^2 + s}$$

We discretize this controller using Tustin to get

$$C_T(z)\frac{594\,z^2 - 1093\,z + 502.8}{z^2 - 0.6486\,z + 0.3514}$$



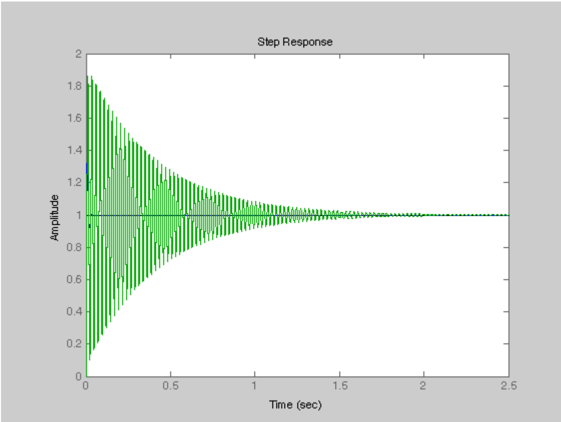
The bode plots of continuous and discrete closed loops and the step responses are shown in the adjacent figure.

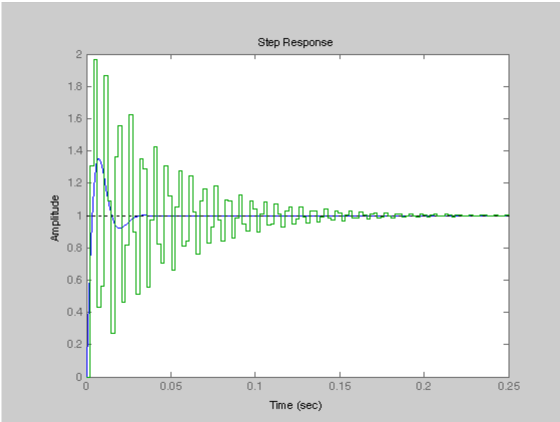
3. In this part, we examine how the loop behavior changes if we change the sampling rate but keep the discrete PID transfer function the same. One interpretation of this is a non-real time implementation where the sampling time can vary. To study this we must form the loop between the plant H discretized at the new frequency and the PID with a sample time adjustment. This is a little involved since MATLAB checks for consistency in sample times. The relevant commands are:

```
tsn=1.6e-4,Ct.Ts=tsn;step(feedback(H*C,1),feedback(c2d(H,tsn)*Ct,1))
```

Then, change tsn and repeat.

The limits of stability are approximately one decade of sampling rates around the design value (1e-3):
3.5e-4 < Ts < 2.1e-3.
(Roughly a factor of 2-3; however, the practical limits for acceptable performance are much tighter.)
The responses are shown below for Ts=3.5e-4 (left) and 2.1e-3 (right).





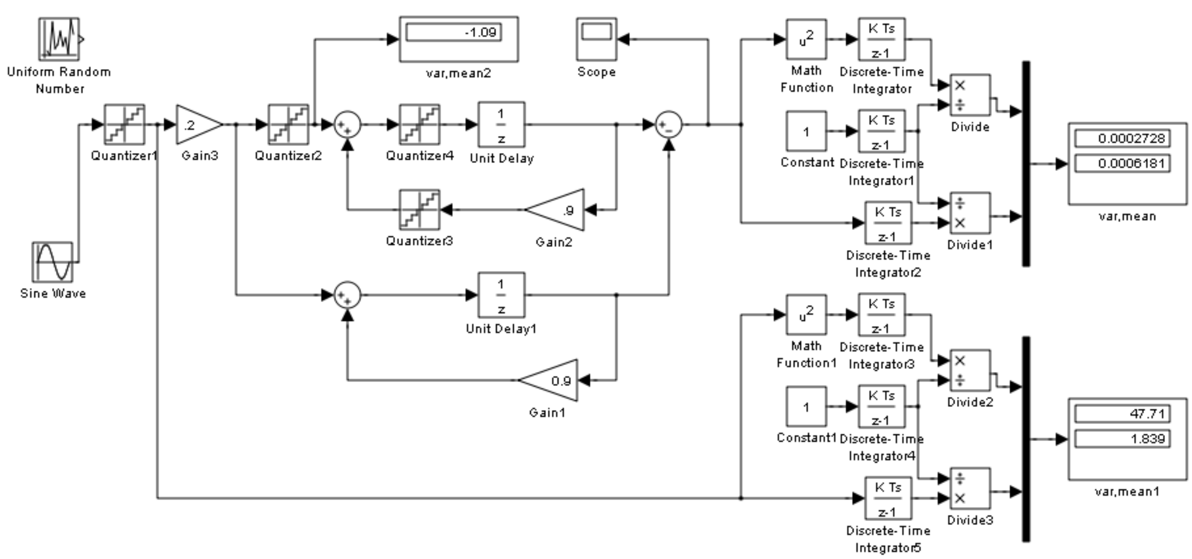
P.2

Consider the system $x(k+1) = 0.9x(k) + 0.2u(k)$, where the multiplications and the addition are quantized to 0.01. Use simulation to assess the mean, worst-case amplitude, and variance of the error due to quantization (compared to non-quantized operation). Apply various inputs u(k), e.g., random, sinusoid, quantized to 0.01. Compare your results with the theoretical bounds computed from the corresponding transfer functions.

1. mean(x_n) = G(1)mean(n)
2. max|x_n(k)| <= sum|g(k)|max|n(k)|, (g = Z^{-1}{G})
3. var{x_n(k)} <= $|G(e\text{^}j\Omega)|_2^2$var{n(k)}
4. RMS bound: var{x}~RMS^2{x} <= $\max_\Omega|G(e\text{^}j\Omega)|^2$ RMS^2{n}
5. Use MATLAB's *"linmod"* command to generate the desired transfer functions directly from Simulink models.

The quantizations are modeled as random noise n(k) of uniform distribution ½ LSB. The system now is $x(k+1) = 0.9x(k) + 0.2u(k) + n_1(k) + n_2(k) + n_3(k)$ .Then, the output contribution of that noise (x_n(k)) is described by the transfer function G(z)=1/(z-0.9) (the quantization occurs after taking the product 0.5u(k)) and is bounded as follows:
1. mean(x_n) = G(1)mean(n1)+ G(1)mean(n2) + G(1)mean(n3) ;
2. |x_n(k)| <= sum|g(k)| 3max|n(k)|, (g = Z^{-1}{G})
3. var{x_n(k)} <= $|G(e\text{^}j\Omega)|_2^2$var{n1(k)} +$|G(e\text{^}j\Omega)|_2^2$var{n2(k)} +$|G(e\text{^}j\Omega)|_2^2$var{n3(k)}
4. RMS bound: var{x}~RMS^2{x} <= $\max_\Omega|G(e\text{^}j\Omega)|^2$ RMS^2{n}

Notice that all quantization noises enter at the same node so there is only one transfer function. This is a coincidence of the model structure. Cascade models will not enjoy such a property.



Computing the theoretical estimates, G(1)=1/0.1=10. sum(|g(k)|=10). $|G|_2^2$ = sum$|g(k)|^2$=1/(1-0.9^2)=5.263. (The last one is based on Parseval, or simply using Matlab.) $\max_\Omega|G(e\text{^}j\Omega)|^2$=100.

For a round-off quantization, whose mean is 0 LSB, max(|n|)= ½ LSB = 0.005 and var = $(½\ LSB)^2/^3$ =8.33e-6. Thus,
1. mean(x_n) = 10*(0+0+0) = 0
2. |x_n(k)| <= 10(0.015) = 0.15
3. var{x_n(k)} <= 5.263*(3*8.33e-6) = 0.13e-3
4. RMS bound: var{x}~RMS^2{x} <= 100 *(3*8.33e-6) = 2.5e-3

The Simulink implementation of a simulator is shown above. We try different input signals: random numbers (uniform or gaussian), sinusoids, etc. In general, we must try different frequencies and different amplitudes as well, since the quantization makes the system nonlinear and the response to a scaled input is not simply the scaled response. The main results are tabulated below. Notice that, for the large stochastic input, the variance estimate is the least conservative one, but the estimate does not bound the actual signal. For low amplitude excitation where the nonlinearity is more prevalent, the deterministic bounds are more meaningful.

A priori estimate	Computed rand{-0.05,0.05}	Computed rand{-10,10}	Computed 0.05sin(0.001k)	Computed 10sin(0.001k)
var = 0.13e-3 = 2.5e-3 (rms)	0.65e-3	0.10e-3	2.1e-3	0.29e-3
Max = 0.15	0.063	0.032	0.09	0.073
mean = 0	-0. 12e-3	4.3e-3	0.28e-3	0.033e-3

Also, it is possible that the mean is nonzero. In this case, the variance estimate should be adjusted (for simplicity, we skip the adjustment, keeping track of the mean to correctly interpret the results). For example, the mean estimate becomes relevant if we use a different quantization scheme (floor, ceil) that have a nonzero mean. This estimate is fairly accurate.

P.3

Ziegler-Nichols Tuning: Apply the two Z-N methods to tune a PID for the plants

$$P_1(s) = \frac{(-0.1s+1)}{s^2+4s+1} \qquad P_2(s) = \frac{(-0.5s+1)}{s^2+0.5s+1}\ .$$

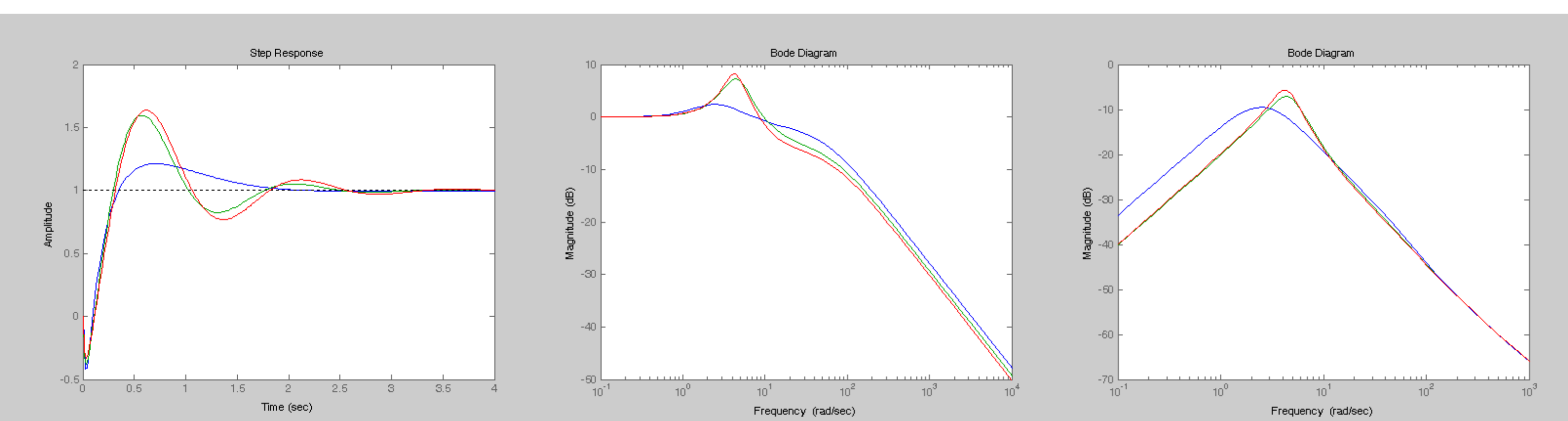
Compare the results with a PID designed for a comparable gain crossover frequency and 50deg. phase margin.

*Hint: Define P as a transfer function object and use step(P) to get an estimate of R,L for the first Z-N tuning. Then iterate k on step(feedback(k*P,1)) until the system is marginally stable (slowly increasing or slowly decreasing response). Then estimate Ku,Pu for the second Z-N tuning. Define the compensators and compare step responses and bode plots for the transfer functions command-to-output and input disturbance-to-output*

```
P=tf([-.1 1],[1 4 1])
figure(1),step(P),grid
R=(.407-.113)/(2.33-.8),L=0.25, %from graph
s=tf([1 0],1);
Kp=1.2/R/L;Ki=0.6/R/L/L;Kd=0.6/R;ZN1=Kp+Ki/s+Kd*s/(.01*s+1)
figure(2),step(fbk(P*5,1))
figure(2),step(fbk(P*20,1))
figure(2),step(fbk(P*40,1))
Ku=40,Pu=6.47-5.49,
Kp=0.6*Ku;Ki=1.2*Ku/Pu;Kd=0.075*Ku*Pu;ZN2=Kp+Ki/s+Kd*s/(.01*s+1)
figure(3),bode(P*ZN1,P*ZN2)
wc=4.65;
[m,p]=bode(P*1/s/(.01*s +1),wc)
pz=-130-(p-360)
tau=tan(pz/2*pi/180)/wc
C=tf(conv([tau 1],[tau 1]),[.01 1 0])
K=1/bode(P*C,wc)
C=tf(conv([tau 1],[tau 1]),[.01 1 0])*K
figure(4),bode(P*C,P*ZN1,P*ZN2)
figure(5),step(fbk(P*C,1),fbk(P*ZN1,1),fbk(P*ZN2,1))
figure(6),bodemag(fbk(P*C,1),fbk(P*ZN1,1),fbk(P*ZN2,1))
figure(7),bodemag(fbk(1,P*C),fbk(1,P*ZN1),fbk(1,P*ZN2))
figure(8),bodemag(fbk(P,C),fbk(P,ZN1),fbk(P,ZN2))
```

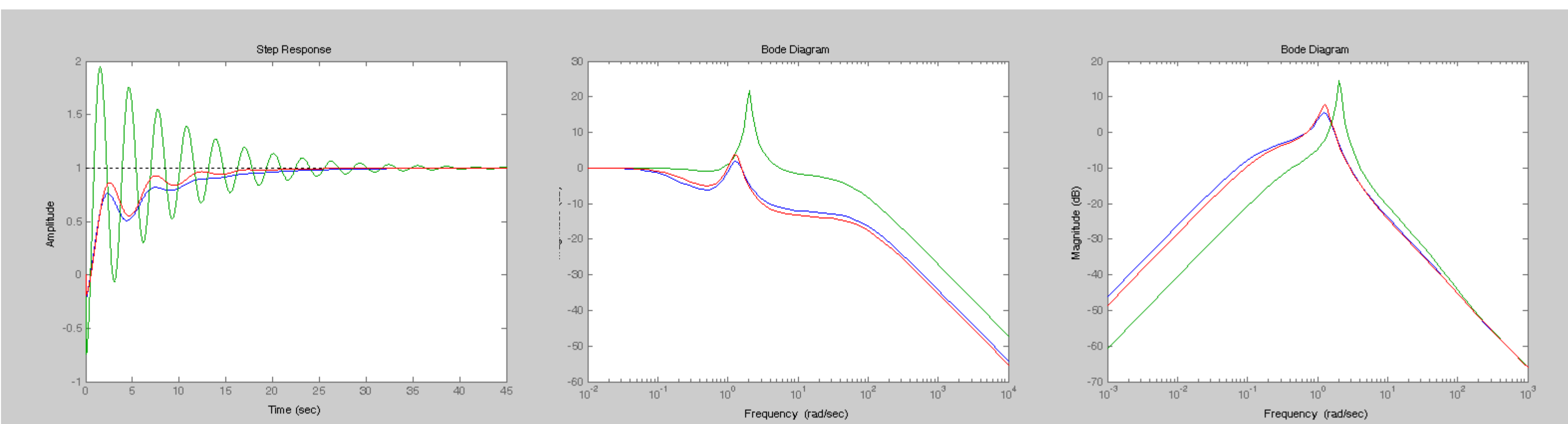
Notice that both ZN methods yield much smaller phase margins than the classical design (20-30 deg). They do, however, offer smaller sensitivity at low frequencies without increasing the loop bandwidth too much. (They do increase the Sensitivity peak and resonance effect).

The step responses, command frequency responses, and input disturbance frequency responses are shown below.



```
P=tf([-.5 1],[1 .5 1])
figure(1),step(P),grid
R=(.541-.15)/(1.7-1.14),L=0.9, %from graph
s=tf([1 0],1);
Kp=1.2/R/L;Ki=0.6/R/L/L;Kd=0.6/R;ZN1=Kp+Ki/s+Kd*s/(.01*s+1)
figure(2),step(fbk(P*5,1))
figure(2),step(fbk(P*.9,1))
figure(2),step(fbk(P*1,1))
Ku=1,Pu=33.8-29.3,
Kp=0.6*Ku;Ki=1.2*Ku/Pu;Kd=0.075*Ku*Pu;ZN2=Kp+Ki/s+Kd*s/(.01*s+1)
figure(3),bode(P*ZN1,P*ZN2)
wc=1.2;
[m,p]=bode(P*1/s/(.01*s +1),wc)
pz=-130-(p-360)
tau=tan(pz/2*pi/180)/wc
C=tf(conv([tau 1],[tau 1]),[.01 1 0])
K=1/bode(P*C,wc)
C=tf(conv([tau 1],[tau 1]),[.01 1 0])*K
figure(4),bode(P*C,P*ZN1,P*ZN2)
figure(5),step(fbk(P*C,1),fbk(P*ZN1,1),fbk(P*ZN2,1))
figure(6),bodemag(fbk(P*C,1),fbk(P*ZN1,1),fbk(P*ZN2,1))
figure(7),bodemag(fbk(1,P*C),fbk(1,P*ZN1),fbk(1,P*ZN2))
figure(8),bodemag(fbk(P,C),fbk(P,ZN1),fbk(P,ZN2))
```

Here the PIDs have difficulty balancing the fast phase transition around the resonance and the bandwidth limitation from the RHP zero. The open-loop step-response ZN is too optimistic and yields a very oscillatory design. For the classical PID we choose the closed-loop ZN crossover. (The one corresponding to the open-loop ZN, has slow and fast modes and is not shown here). The closed-loop ZN and the classical design have similar behavior.



EEE 481 Homework 3

.1

1. Ziegler-Nichols Tuning: Apply the two Z-N methods from the notes to tune a PID for the plants:

$$P_1(s) = \frac{(-0.1s+1)}{s^2+4s+2} \qquad P_2(s) = \frac{10(-0.5s+1)}{s^2+10s+20} \ .$$

2. Compare the results with a PID designed for a gain crossover frequency of 2 rad/s and 45deg. phase margin.

*Hint: Define P as a transfer function object and use step(P) to get an estimate of R,L for the first Z-N tuning. Then iterate k on step(feedback(k*P,1)) until the system is marginally stable (slowly increasing or slowly decreasing response). Then estimate Ku,Pu for the second Z-N tuning. Define the compensators and compare step responses and bode plots for the transfer functions command-to-output and input disturbance-to-output*

.2

1. Design a PID controller to achieve a bandwidth of 1Hz, 50deg phase margin, and to be discretized with a sampling rate of 10Hz for the system with transfer function

$$P\ (s) = \frac{(-0.1s+1)}{s^2+4s+2}$$

2. Compare the results with a design in discrete time directly, where the plant is discretized and the parameters of a discrete-time PID are calculated to achieve the same specifications.

.3

The read arm on a computer disk drive has transfer function $H(s) = \dfrac{1000}{s^2}$.

1. Design an analog PID controller to achieve a bandwidth of approx. 100Hz with 50deg phase margin.
2. Design a discrete PID for the same bandwidth and phase margin, with a sampling rate 1kHz and simulate the closed loop step response.
3. What is the maximum bandwidth that can be achieved with a PID having 50deg phase margin and 1kHz sampling?
4. Suppose that the sampling time clock is unreliable and fluctuates. Test the robustness of the PID in 2 by finding (through simulations) the limits of stability to changes in the sampling rate (i.e., keep the discrete-time PID coefficients the same).
5. Design a simple prefilter to achieve overshoot to step reference changes under 5%.

Hint: You need a complete PID for this problem (2-zeros). You may or may not use a filter for the pseudo-differentiator; if so, choose T = 0.001, consistent with the 1ms sampling time.

EEE 481 Homework 3 SOLUTIONS

.1

1. Ziegler-Nichols Tuning: Apply the two Z-N methods from the notes to tune a PID for the plants:

$$P_1(s) = \frac{-0.1s + 1}{s^2 + 4s + 2} \quad P_2(s) = \frac{10(-0.5s + 1)}{s^2 + 10s + 20}$$

2. Compare the results with a PID designed for a gain crossover frequency of 2 rad/s and 45deg. phase margin.

*Hint: Define P as a transfer function object and use step(P) to get an estimate of R,L for the first Z-N tuning. Then iterate k on step(feedback(k*P,1)) until the system is marginally stable (slowly increasing or slowly decreasing response). Then estimate Ku,Pu for the second Z-N tuning. Define the compensators and compare step responses and bode plots for the transfer functions command-to-output and input disturbance-to-output*

We begin with the step response of the system P1 and estimate the parameters R=0.25,L=0.2 and enter them in the ZN computation:

```
C1a =
2 s^2 + 24 s + 60
   -----------------
           s
```

A similar computation for P2 yields R=0.45, L=1 and

```
C2a=
1.111 s^2 + 2.667 s + 1.333
--------------------------
              s
```

Applying the second method we compute closed-loop step responses with increasing gains until sustained oscillations are obtained. In practice, one should avoid stressing the plant too much and observing few cycles of a slowly decaying oscillation is sufficient to obtain good estimates of the ultimate gain and period.

For P1 we find Ku=38, Pu=1.59-0.595=1, for which

```
C1b=
2.85 s^2 + 22.8 s + 45.6
------------------------
             s
```

For P2 we find Ku=1.8, Pu=1, for which

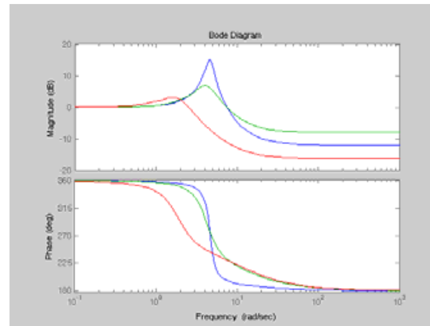
```
C2b=
0.135 s^2 + 1.08 s + 2.16
-------------------------
              s
```

Next, we compute PID gains for both systems using a classical phase margin approach:
Angle{PID} = -180+P.M.-angle{P} at the crossover frequency
Where angle{PID} = N*angle{s+a}-90 and N = 2 for PID, 1 for PI and 0 for I controllers.

(In case a pseudo differentiator is used, a third term is added –angle{Ts+1}).

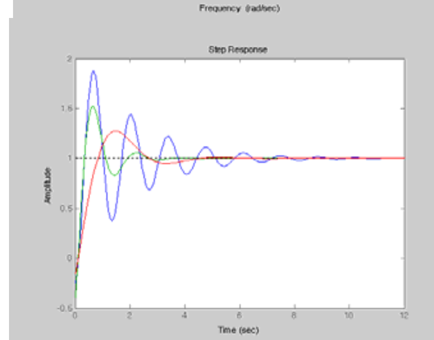
For P1, angle(P1) = -115, so N*angle{s+a} = 70 which is barely achievable with N =1. To remain consistent with the rest of the tunings, we use N=2, from which **a = 2.86.**
Entering this value in the gain computation we find
C1c=tf(conv([1 2.86],[1 2.86]),[1 0])
[m,p]=bode(P1*C1c,2), **m=0.75**
C1c=tf(conv([1 2.86],[1 2.86]),[1 0])/m
C1c=

```
1.328 s^2 + 7.595 s + 10.86
---------------------------
             s
```



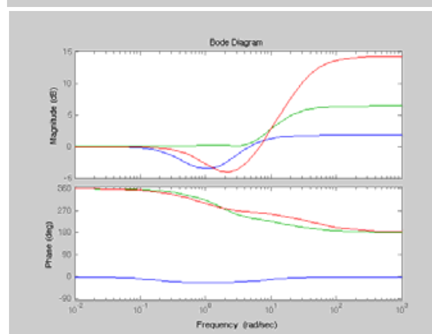
For P2, angle(P2) = -96, so N*angle{s+a} = 51 which is achievable with N =1 or 2. Preserving the choice N=2, we get **a = 4.2.** Entering this value in the gain computation we find
C2c=tf(conv([1 4.2],[1 4.2]),[1 0])
[m,p]=bode(P2*C2c,2), **m=6.0**
C2c=tf(conv([1 2.86],[1 2.86]),[1 0])/m
C2c=

```
0.1674 s^2 + 0.9574 s + 1.369
-----------------------------
              s
```

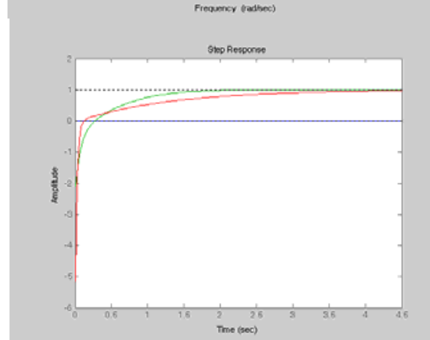


We now use all these controllers to compute step and frequency responses:

C1a (blue) is very oscillatory but yields similar bandwidth as C1b (green), which produces a fairly reasonable response. C1c (red) is a lower bandwidth controller but has also good response characteristics.



On the other hand, C2a is unstable and both C2b and C2c exhibit large inverse response and amplify high frequencies. The problem here is in the right half plane zero that is not properly attenuated by Z-N and is very near the crossover frequency in the classical design. Remedies for these problems include the addition of a lowpass filter and the reduction of the crossover frequency.



.2

1. Design a PID controller to achieve a bandwidth of 1Hz, 50deg phase margin, and to be discretized with a sampling rate of 10Hz for the system with transfer function

$$P_1(s) = \frac{-0.1s + 1}{s^2 + 4s + 2}$$

2. Compare the results with a design in discrete time directly, where the plant is discretized and the parameters of a discrete-time PID are calculated to achieve the same specifications.

The angle condition for this problem is

$$angle\{P_1\} + angle\{PID\} + angle\{ZOH\} = -180 + 50$$

evaluated at 1Hz = 6.28rad/sec. First, we estimate the crossover value as 6.28/1.5 = 4.2rad/sec. The ZOH discretized at 10Hz (T=0.1sec) has angle –wT/2 = 0.21rad = -12deg. The plant P1 has angle [m,p]=bode(P1,4.2); p = - 156deg.
Hence, angle{PID} = 38 deg. This (positive phase) can only be achieved by a PID. Furthermore, since we are trying to design a controller for a plant with roll-off rate only -20db/dec and the sampling rate is not very high relative to the desired bandwidth, it is advisable to include a pseudo-differentiator pole (sT/2+1) in the PID. If not, then the controller discretization will not approximate well the continuous design. Moreover, it will have no high frequency roll-off and will be susceptible to high frequency noise. For consistency in our comparison, we select this pole to be at 2/T so that its Tustin discretization will be simply “z”. We will make the same choice later in the discrete design.
Adding the pseudo differentiator pole will add lag in the PID. We now have

$$angle\{PID\} = N\tan^{-1}\frac{4.2}{a} - 90 - \tan^{-1}\frac{T4.2}{2} = N\tan^{-1}\frac{4.2}{a} - 102 = 38$$

The only choice for N is 2, for which we find

$$a = \frac{4.2}{\tan(\frac{140}{2})} = 1.53$$

Next, we compute the controller gain from the magnitude equation

```
>> C=tf(conv([1 1.53],[1   1.53]),[0.05 1 0]);
>> [m,p]=bode(P1*C,4.2); m =   2.2000e-001
>> C=tf(conv([1 1.53],[1   1.53]),[0.05 1 0])/m
Transfer function:
4.545 s^2 + 13.91 s + 10.64
---------------------------
          0.05 s^2 + s

Cd=c2d(C,0.1,'tustin')

Transfer function:
52.67 z^2 - 90.38 z + 38.77
---------------------------
              z^2 - z

Sampling time: 0.1
```

The closed-loop response is reasonable and the loop has the desired 50deg phase margin but the closed loop bandwidth is large, 13.7 rad/sec instead of 6.28. If there is a strict requirement on the closed loop bandwidth, then we can apply an iterative algorithm (similar to the solution of nonlinear equations) to find the crossover frequency that will produce the desired bandwidth. Such algorithms may or may not converge quickly to the solution and it is highly recommended that the intermediate steps are automated for their application.

Next, we will perform the design entirely in discrete time. Here we consider the PID of the form

$$C(z) = K\frac{(z-a)^2}{z(z-1)}$$

while the plant is Pd = c2d(P1,0.1)

Transfer function:
-0.003827 z + 0.01206
----
z^2 - 1.654 z + 0.6703

Sampling time: 0.1

We evaluate its phase at the crossover frequency 4.2 rad/sec = 0.42 rad/sample (Note: Matlab assumes the frequency is expressed in rad/sec)

[m,p]=bode(Pd,4.2)

p = -1.6815e+002

$$angle\{PID\} = N\tan^{-1}\frac{\sin 0.42}{\cos 0.42 - a} - 0.42(rad) - \tan^{-1}\frac{\sin 0.42}{\cos 0.42 - 1} = -180 + 50 + 168$$

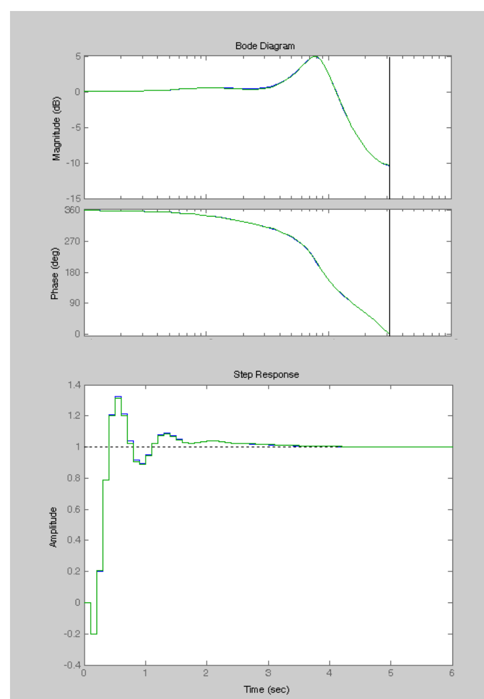
$$\Rightarrow N\tan^{-1}\frac{\sin 0.42}{\cos 0.42 - a} - 24 - (-78 + 180) = 38 \Rightarrow 2\tan^{-1}\frac{\sin 0.42}{\cos 0.42 - a} = 164$$

$$\Rightarrow \mathrm{a} = 0.86$$

Thus,

```
>> D=tf(conv([1 -a],[1 -a]),[1 -1 0],.1);
>> [m,p]=bode(Pd*D,4.2)
m =     1.8962e-002
>> D=tf(conv([1 -a],[1 -a]),[1 -1 0],.1)/m
```

Transfer function:
52.74 z^2 - 90.71 z + 39
----
z^2 - z



The two controllers (continuous design-compensated-Tustin-discretized and full discrete) are virtually identical as illustrated by their closed loop step and frequency responses.

.3

The read arm on a computer disk drive has transfer function

$$H(s) = \frac{1000}{s^2}$$

1. 1. Design an analog PID controller to achieve a bandwidth of approx. 100Hz with 50deg phase margin.
2. 2. Design a discrete PID for the same bandwidth and phase margin, with a sampling rate 1kHz and simulate the closed loop step response.
3. 3. What is the maximum bandwidth that can be achieved with a PID having 50deg phase margin and 1kHz sampling?
4. 4. Suppose that the sampling time clock is unreliable and fluctuates. Test the robustness of the PID in 2 by finding (through simulations) the limits of stability to changes in the sampling rate (i.e., keep the discrete-time PID coefficients the same).
5. 5. Design a simple prefilter to achieve overshoot to step reference changes under 5%.

Hint: You need a complete PID for this problem (2-zeros). You may or may not use a filter for the pseudo-differentiator; if so, choose T = 0.001, consistent with the 1ms sampling time.

1. Analog PID design: Crossover frequency = 100*6.28/1.5 = 420 rad/sec.

$$C(s) = K\frac{(s+a)^2}{s(Ts+1)}$$

1.1 Angle equation

$$angle\{H(s)\} + angle\{C(s)\} = -180 + 50 \Rightarrow angle\{C(s)\} = 50$$

$$angle\{C(s)\} = 2\tan^{-1}\frac{420}{a} - 90 - \tan^{-1}0.42 = 50 \Rightarrow \tan^{-1}\frac{420}{a} = 81 \Rightarrow a = 64$$

1.2. Gain equation

```
C=tf(conv([1 64],[1 64]),[.001 1 0])
[m,p]=bode(P*C,420)

m =

  2.2462e+000
C=tf(conv([1 64],[1 64]),[.001 1 0])/m

Transfer function:
0.4452 s^2 + 56.99 s + 1824
---------------------------
        0.001 s^2 + s
```

The feedback loop for this controller has bandwidth 113Hz, PM 50deg and a 30% overshoot (quite reasonable). However, the controller zeros are placed at 64rad/sec, contributing 81degrees of phase lead each, near the limit of design feasibility. The fact that the zeros are so much lower than the crossover frequency means that the closed-loop will have some slow pole-zero cancellations (compared to bandwidth) that will take extra time to dissipate.

2. Discrete PID design

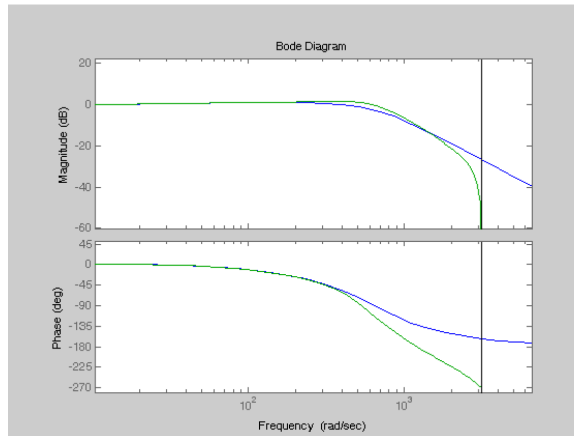
2.1. Angle equation: Need additional phase lead to compensate for the ZOH, -wT/2 = 0.21 rad/sec = 12 deg.

$$angle\{C(s)\} = 2\tan^{-1}\frac{420}{a} - 90 - \tan^{-1}0.42 = 50 + angle\{ZOH\} \Rightarrow a = 22$$

The situation is worse now, with the zeros even lower. Possible remedies for this situation are to decrease the time constant of the pseudo-derivative term (e.g. T/2) or increase the sampling frequency. At any rate, we will continue according to the problem statement.

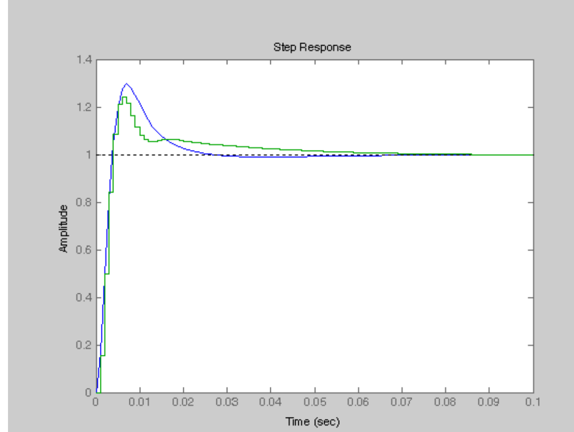
2.2. Gain equation

```
C=tf(conv([1 22],[1 22]),[.001 1 0])
[m,p]=bode(P*C,420)

m =
   2.2012e+000
C=tf(conv([1 22],[1 22]),[.001 1 0])/m

Transfer function:
0.4543 s^2 + 19.99 s + 219.9
----------------------------
         0.001 s^2 + s
```



2.3. Controller discretization

```
Cd3=c2d(C,0.001,'tustin')

Transfer function:
309.6 z^2 - 605.7 z + 296.2
---------------------------
   z^2 - 1.333 z + 0.3333

Sampling time: 0.001
```



The response plots show that even though the discrete time implementation is at the limits of its capabilities, the match between continuous and discrete responses is reasonably close.

3. The maximum achievable bandwidth for either continuous or discrete design is determined from the angle equation. In the continuous case, the limiting factor is the angle of the pseudo-derivative term (since T is fixed). The plant here has fixed phase for all frequencies, which is rather uncommon. The maximum the zeros can contribute is 90 deg (practical limit is 75-80deg). Thus,

$$angle\{C(s)\} = 180 - 90 - \tan^{-1}w * 0.001 = 50 \Rightarrow \tan^{-1}w * 0.001 = 40 \Rightarrow w_{max} = 839 rad/s$$

For the discrete design, the ZOH contributes additional phase lag:

$$angle\{C(s)\} = 180 - 90 - \tan^{-1}w * 0.001 = 50 + wT/2 \Rightarrow \frac{w}{2000} + \tan^{-1}\frac{w}{1000} = 40 \Rightarrow w_{max} = 488 rad/s$$

(We find the solution of this nonlinear equation by plotting the values of the left hand-side and finding the crossing.)

Note: In general, for such problems a pure discrete PID design might do a little better since it does not involve any approximations.

4. Iterate for different T the evaluation of closed-loop step responses. In order to maintain the same discrete controller parameters we need to extract its numerator and denominator, as follows:

```
[num,den]=tfdata(Cd3,'v')
num =
   3.0956e+002 -6.0565e+002   2.9624e+002
den =
   1.0000e+000 -1.3333e+000   3.3333e-001

T=0.001
step(fbk(P*C,1),fbk(c2d(P,T)*tf(num,den,T),1),t)
T= … etc
```

For increasing sampling times the limit is T*2. For decreasing sampling times the limit is T/7.

5.We use the closed-loop Bode plot for guidance to place the prefilter pole.
We begin with a value around 100rad/sec and iterate the step response evaluation until the overshoot drops below the required threshold. The zero we can fix roughly around bandwidth where the frequency response is rolling off.

```
step(c2d(tf([1/1000 1],[1/200 1]),.001,'tustin')*fbk(c2d(P,0.001)*Cd,1),fbk(c2d(P,0.001)*Cd,1))
```

Note that looking at step responses provides better resolution for the overshoot than looking at Bode plots.

The required prefilter to satisfy the overshoot specification is $F(s) = \frac{\frac{1}{1000}s+1}{\frac{1}{200}s+1}$ and its discrete time implementation is (using Tustin)

```
Transfer function:
0.2727 z - 0.09091
------------------
      z - 0.8182

Sampling time: 0.001
```

Pr. 24 $\quad E(s) = \dfrac{2(1-e^{-5s})}{s(s+2)}$; $T = 1$ sec

$$e(t) = \mathcal{L}^{-1}\{E(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s(s+2)}\right\} - \mathcal{L}^{-1}\left\{\frac{2e^{-5s}}{s(s+2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s(s+2)}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s(s+2)}\right\}\Big|_{t \leftarrow t-5}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{-1}{s+2}\right\} = \left(1-e^{-2t}\right)u(t)$$

$$\Rightarrow \quad e(t) = \left(1-e^{-2t}\right)u(t) - \left(1-e^{-2(t-5)}\right)u(t-5)$$

Sample at $\quad t_k = kT = k$

$$\Rightarrow \quad e_k = e(t_k) = \left(1-e^{-2k}\right)u(k) - \left(1-e^{-2(k-5)}\right)u(k-5)$$

Let $\lambda = e^{-2}$. Then

$$Z\{e_k\} = Z\{(1-\lambda^k)u(k)\}\left(1-z^{-5}\right)$$

$$= \left(\frac{z}{z-1} - \frac{z}{z-\lambda}\right)\left(1-z^{-5}\right)$$

$$= (1-\lambda)\,\frac{z^5-1}{z^4(z-1)(z-\lambda)}$$

Pr. 2.11

$$\frac{Y(z)}{E(z)} = \frac{1}{z^2 - \frac{3}{4}z + \frac{1}{8}} = \frac{1}{(z-\frac{1}{2})(z-\frac{1}{4})}$$

$$E(z) = z^{-1}\left(\frac{z}{z-1}\right) = \frac{1}{z-1}$$

$$\Rightarrow \; Y(z) = \frac{1}{(z-\frac{1}{2})(z-\frac{1}{4})(z-1)} = \frac{-8}{z-\frac{1}{2}} + \frac{16/3}{z-\frac{1}{4}} + \frac{8/3}{z-1}$$

$$\Rightarrow \; y_k = \mathcal{Z}^{-1}\{Y(z)\} = -8\left(\tfrac{1}{2}\right)^{k-1}u(k-1) + \tfrac{16}{3}\left(\tfrac{1}{4}\right)^{k-1}u(k-1) + \tfrac{8}{3}u(k-1)$$

Evaluating, :

$$y_0 = 0$$
$$y_1 = -8 + \tfrac{16}{3} + \tfrac{8}{3} = 0$$
$$y_2 = -8(\tfrac{1}{2}) + (\tfrac{16}{3})(\tfrac{1}{4}) + (\tfrac{8}{3}) = 0$$
$$y_3 = -8(\tfrac{1}{4}) + (\tfrac{16}{3})(\tfrac{1}{16}) + \tfrac{8}{3} = 1$$
$$y_4 = -8(\tfrac{1}{8}) + (\tfrac{16}{3})(\tfrac{1}{64}) + \tfrac{8}{3} = \tfrac{7}{4}$$

Alternatively, performing the recursion with $y_0 = y_1 = 0$

$$y_2 = e_0 + \tfrac{3}{4}y_1 - \tfrac{1}{8}y_0 = 0$$
$$y_3 = e_1 + \tfrac{3}{4}y_2 - \tfrac{1}{8}y_1 = 1$$
$$y_4 = e_2 + \tfrac{3}{4}y_3 - \tfrac{1}{8}y_2 = 1 + \tfrac{3}{4} = \tfrac{7}{4}$$

Pr 2.26

$$x_{k+1} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u_k$$

$$y_k = (-2, 1)\, x_k$$

$$G(z) = C(zI-A)^{-1}B = (-2,1)\begin{pmatrix} z & -1 \\ 0 & z-3 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{z(z-3)}(-2,1)\begin{pmatrix} z-3 & 1 \\ 0 & z \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{-z+4}{z(z-3)}$$

Coordinate transformation: take $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which simply swaps the two states. Then $P^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$P^{-1}AP = \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix}, \quad P^{-1}B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad CP = (1, -2)$$

The numerical verification of the various properties is now straightforward.

Pr 2.31

$$x_{k+1} = \begin{pmatrix} 1.1 & 1 \\ -0.3 & 0 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u_k$$

$$x_k = \Phi_k x_0 + \sum_{0}^{k-1} \Phi_{k-1-j} B u_j$$

When $x_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $u_k = 0$, $x_k = \Phi_k x_0 = \begin{pmatrix} 1.1 & 1 \\ -0.3 & 0 \end{pmatrix}^k \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

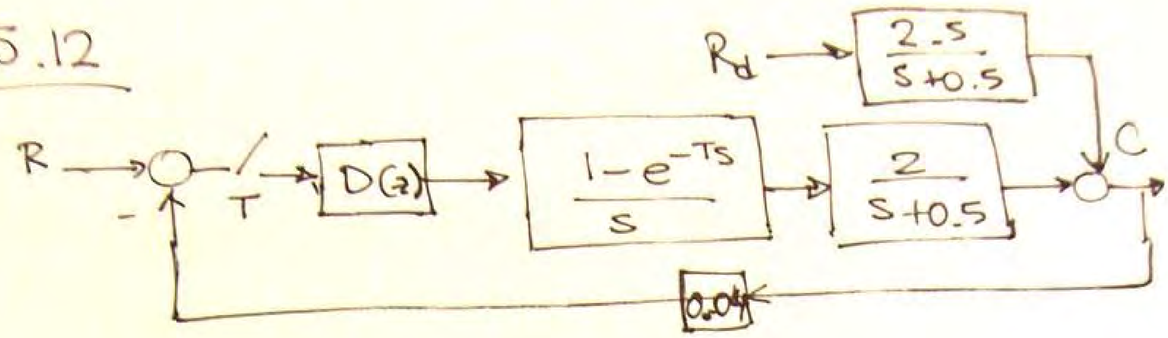
$$\Phi_0 = I, \quad \Phi_1 = \begin{pmatrix} 1.1 & 1 \\ -0.3 & 0 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} 0.91 & 1.1 \\ -0.33 & -0.3 \end{pmatrix}, \dots$$

$$\Rightarrow x_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad x_1 = \begin{pmatrix} 0.9 \\ 0.3 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1.29 \\ -0.27 \end{pmatrix}, \dots$$

Also, $y_k = C x_k = (1, -1)\, x_k \Rightarrow y_0 = -3,\ y_1 = 0.6,\ y_2 = 1.56$

...

Pr. 5.12



Plant T.F.: Sample $\mathcal{L}^{-1}\left\{ \frac{1-e^{-Ts}}{s} \frac{2}{s+0.5} \right\}$ and take z-transform. Since the sampling occurs at $t_k = kT$, same as the ZOH, the same result is obtained by computing $\mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{ \frac{2}{s(s+0.5)} \right\}\Big|_{t_k} \right\} \cdot (1-z^{-1})$.

This is also C2D in Matlab with the ZOH option.

$$\mathcal{L}^{-1}\left\{ \frac{2}{s(s+0.5)} \right\} = \mathcal{L}^{-1}\left\{ \frac{4}{s} + \frac{-4}{s+0.5} \right\} = 4u(t)\left(1-e^{-0.5t}\right)$$

Sample: $4u(k)(1-\lambda^k)$; $\lambda = e^{-0.5T}$

z-transform $\mathcal{Z}\{4u(k) - 4u(k)\lambda^k\} = 4\frac{z}{z-1} - 4\frac{z}{z-\lambda}$

$$\Rightarrow G_{dt}(z) = 4z\frac{1-\lambda}{(z-1)(z-\lambda)} \cdot (1-z^{-1}) = \frac{4(1-\lambda)}{z-\lambda}.$$

The input at $D(z)$ is $R(z) - \bar{C}(z)$, $r_k, \bar{c}_k$ being the sampled reference and sensed-output signals ($\bar{C}_k = 0.04\, C_k$).

Then, $C(z) = G_{dt}(z)\, D(z) \left[R(z) - 0.04\, C(z) \right]$

$$\Rightarrow C(z) = \frac{G_{dt} D}{1 + 0.04\, G_{dt} D} R(z)$$

(here we assumed $R_d = 0$)

or.,
$$\frac{C}{R} = \frac{4(1-\lambda) D}{(z-\lambda) + 0.16(1-\lambda) D}$$

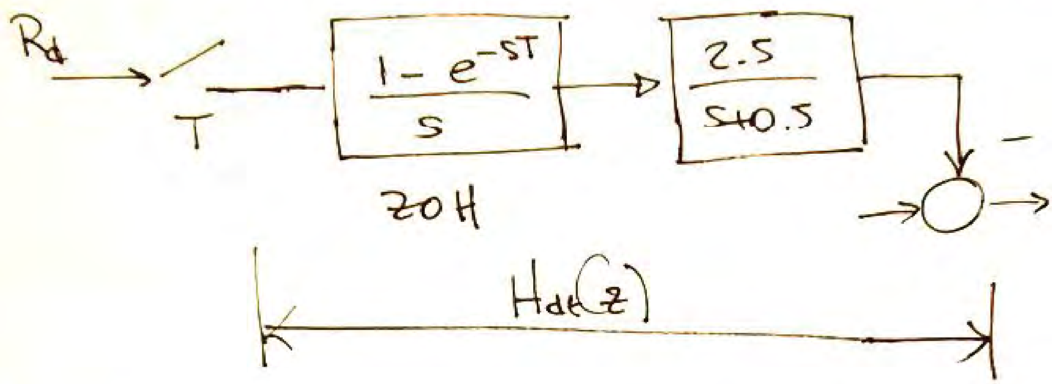
If R_d is not zero,

$$C(s) = \frac{2}{s+0.5} M(s) - \frac{2.5}{s+0.5} R_d(s)$$

so the input to the controller will be the sampled $C(t)$:

$$C(z) = G_{dt}(z)\, D(z) \left[0 - 0.04\, C(z) \right] - \mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{ \frac{2.5}{s+0.5} R_d \right\} \Big|_{t_k} \right\}$$

In general, R_d can be any CT signal and the last computation cannot be simplified. But here, the disturbance is the opening of the door, so we can assume it is a piecewise constant signal and the transitions happen at the sampling times. In such a case we can model the disturbance as



So, $H_{dt}(z) = -\frac{2.5}{2} \, G_{dt}(z) = \frac{-5(1-\lambda)}{z-\lambda}$

Then, $$C(z) = \frac{H_{dt}(z)}{1 + 0.04\, G_{dt}(z)\, D(z)} R_d(z)$$

$$= -\frac{5(1-\lambda)}{(z-\lambda) + 0.16(1-\lambda)D} R_d(z)$$

Using superposition, the total output is

$$C(z) = \frac{4(1-\lambda)D}{(z-\lambda)+0.16(1-\lambda)D} R(z) - \frac{5(1-\lambda)}{(z-\lambda)+0.16(1-\lambda)D} R_d(z)$$

Pr. 8.2 $\quad P(s) = \dfrac{1}{s(s+1)}$

Find ZOH - discrete equivalent (e.g. MATLAB)

$$P_d(z) = \frac{0.368z + 0.264}{z^2 - 1.37z + 0.368}$$

Sketching the Bode plot of P_d we have that the crossover is at $\omega = 0.77$ rad/s where $\angle P_d = -150°$ so $PM = 30°$.

NOTE: Bode of $P_d(z)$ is $|P_d(e^{j\Omega})|$ vs Ω and $\angle P_d(e^{j\Omega})$ vs Ω where Ω is the discrete frequency in rad/sample: It is related to the continuous freq. ω = rad/sec by $\Omega = \omega T$. Matlab plots $|P_d(e^{j\Omega})|$ vs ω to compare easily with the C-T counterparts.

For a $45°$ phase margin, the crossover is at 0.55 rad/sec where $|P_d| = 3.9$ dB, so

$K = -3.9\text{dB} = 0.638.$

Using Matlab to simulate the step response we find that OSH $\simeq 8\%$, $t_R \cong 3.01 - 0.63$
$= 2.38$

Next we design a PI compensator to also have $45°$ PM. The PI will introduce more lag than a pure gain so we need to decrease the crossover frequency to a value where $\angle P_d$ is the expected (or designed) lag from the PI. Let's suppose that this value is $\angle PI(j\omega_c) = 5°$. Then $\angle P_d(j\omega_c)$ must be $-130°$, so $\omega_c = 0.486$ rad/s (from the Bode plot). Now the PI has transfer function

$$C(z) = K \frac{z-a}{z-1}$$

$$\Rightarrow \angle C(e^{j\Omega}) = \angle(e^{j\Omega} - a) - \angle(e^{j\Omega} - 1)$$

$$\Omega = \omega T = \omega = 0.486$$

$$\angle C = -5° = \tan^{-1} \frac{\sin \Omega}{\cos \Omega - a} - \tan^{-1} \frac{\sin \Omega}{\cos \Omega - 1}$$

corrections may be needed

Solving for a :

$$\tan^{-1}\frac{\sin\Omega}{\cos\Omega - a} = \tan^{-1}\frac{\sin\Omega}{\cos\Omega - 1} - 5^{\circ} + \text{CORR}$$

IRRELEVANT after taking tan

$$\Rightarrow \frac{\sin\Omega}{\cos\Omega - a} = \tan\left[\tan^{-1}\frac{\sin\Omega}{\cos\Omega - 1} - 5^{\circ}\right]$$

$$\Rightarrow a = \cos\Omega - \frac{\sin\Omega}{\tan\left[-\tan^{-1}\frac{\sin\Omega}{1-\cos\Omega} - 5^{\circ}\right]}$$

$$= 0.958$$

Calculating $\left| P_d \frac{z-a}{z-1} \right|_{z=e^{j\Omega}} = 5.13 \text{ dB}$

$$\Rightarrow K = -5.13 \text{ dB} = 0.554$$

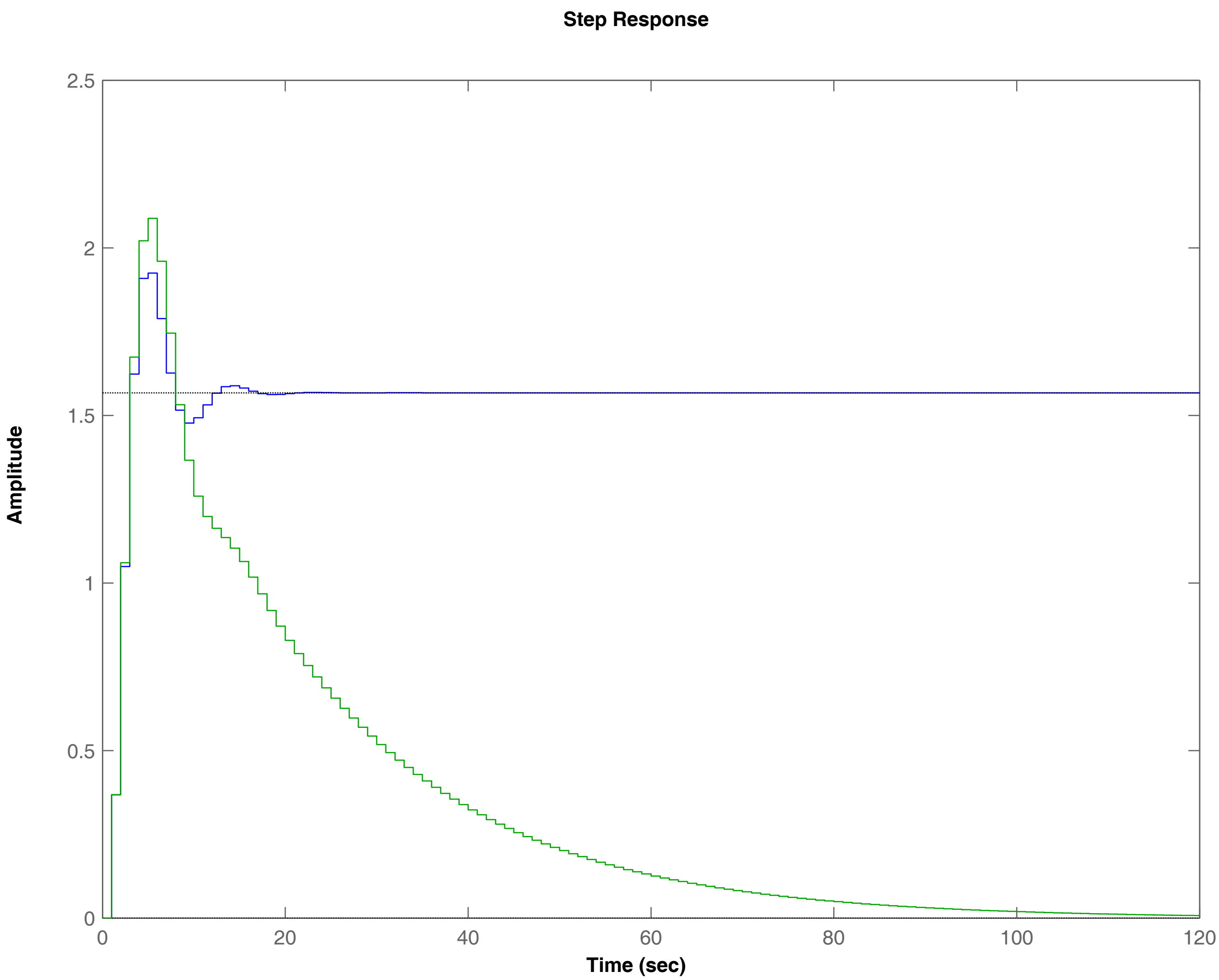
$$\Rightarrow C(z) = \frac{0.554z - 0.531}{z-1}.$$

Simulating the step response we find

$$OSH \simeq 28\%, \quad t_R \simeq 2.38.$$

The step response with this compensator converges slower than before but the big difference is in the response to step input disturbances.

```
P=tf(1,[1 1 0])
uc=tf([-1 1],[1 1])
Pd=c2d(P,1)
bode(P,Pd)
10^(-3.9/20)  % dB value from bode plot figure
step(fbk(.638*Pd,1))  % estimate OSH and Tr
bode(fbk(.638*Pd,1),fbk(.638*P,1))
grid
bode(P,Pd)
grid
W=.486   % frequency value from bode plot figure
a=cos(W)-sin(W)/tan(-pi*5/180-atan(sin(W)/(1-cos(W))))
C=tf([1 -a],[1 -1],1)
bode(C*Pd)
10^(-5.13/20)   % dB value from bode plot figure
C=tf([1 -a],[1 -1],1)*.554
step(fbk(.638*Pd,1),fbk(C*Pd,1))  % estimate OSH and Tr
step(fbk(Pd,.638),fbk(Pd,C))  % see the difference in input disturbance response
```



Step Response
Amplitude
Time (sec)
2.5
2
1.5
1
0.5
0
0
20
40
60
80
100
120

EEE 481 HW2 SOLUTIONS

① a. $\mathcal{Z}\{u(k-1)\} = z^{-1}\mathcal{Z}\{u(k)\} = z^{-1}\frac{z}{z-1} = \frac{1}{z-1}$ $(|z|>1)$

b. $\mathcal{Z}\{\sin\frac{\pi}{10}k\} = \mathcal{Z}\{\sin\frac{\pi}{10}k\,u(k) + \sin\frac{\pi}{10}k\,u(-k-1)\}$

"CAUSAL" RIGHT-SIDED

"ANTI-CAUSAL" LEFT-SIDED

$= X_R(z) + X_L(z)$ (if it exists)

FROM TABLES: $X_R(z) = \frac{\sin\pi/10\, z^{-1}}{1-2\cos\pi/10\, z^{-1}+z^{-2}}$ ROC: $|z|>1$

$X_L(z) = \mathcal{Z}\{\sin\frac{\pi}{10}(-k)\,u(-(-k)-1)\}\Big|_{z^{-1}} = \mathcal{Z}\{-\sin\frac{\pi}{10}k\,u(k-1)\}\Big|_{z^{-1}}$

$= -\frac{\sin\pi/10\, z^{-1}}{1-2\cos\pi/10\, z^{-1}+z^{-2}}\Big|_{z^{-1}} = -\frac{\sin\pi/10\, z^{2}}{1-2\cos\pi/10\, z+z^{2}}$ ROC: $|z|<1$

Note: the left-& right-sided sequences have no common ROC. The z-transform of the original has no ROC (does not converge). Also apparent from the observation that $\sin\pi/10\,k$ is not bounded by an exponential over $\mathbb{Z}$

c. $\mathcal{Z}\{(1-e^{-k})u(k)\} = \frac{z}{z-1} + \frac{-z}{z-e^{-1}}$ $(|z|>1)$

d. $\mathcal{Z}\{1.1^k u(k-1)\} = z^{-1}\cdot\frac{z}{z-1.1} = \frac{1}{z-1.1}$ $(|z|>1.1)$

(2) With zero Initial Conditions, we can simplify the transforms

$\mathcal{Z}\{y(k+2)\} = z^2 \mathcal{Z}\{y(k)\}$ etc.

$$\therefore \; Y(z) = \frac{1}{z^2 - \frac{3}{4}z + \frac{1}{8}} \cdot \frac{1}{z-1} = \frac{8/3}{z-1} + \frac{-8}{z-0.5} + \frac{16/3}{z-0.25}$$

$$\Rightarrow y(k) = \frac{8}{3} u(k-1) - 8 \frac{1}{2^{k-1}} u(k-1) + \frac{16}{3} \frac{1}{4^{k-1}} u(k-1)$$

Verify: $y(0)=0$, $y(-1)=0$,

and $y(1) = 0$, $y(2) = 0$, $y(3) = 1$, ...

From direct substitution on the recursion $y_k = x_{k-2} - \frac{1}{8} y_{k-2} + \frac{3}{4} y_{k-1}$ we obtain the same result.

Note: For initial value problems the general state-space solution may have a simplicity advantage:

Define $X(k) = \begin{pmatrix} y_k \\ y_{k+1} \end{pmatrix}$. Then,

$$X(k+1) = \underbrace{\begin{pmatrix} 0 & 1 \\ -1/8 & 3/4 \end{pmatrix}}_{A} X(k) + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} x(k)$$

$$y(k) = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{C} X(k)$$

I.C. $X(-1) = 0$. Translate to 0 for the standard expression:

$$X_{k+1} = A X_k + B x(k-1)$$
$$= A X_k + B u(k-2)$$

I.C. $X(0) = 0$.

etc

③ a. Starting with $y(s) = G(s)u(s) = \frac{3s+4}{s^2+5s+6} u(s)$,

Define $w(s) = \frac{1}{s^2+5s+6} u(s)$, then $y(s) = (3s+4)\, w(s)$

Write the state space model for w:

$$X = \begin{pmatrix} w \\ \dot{w} \end{pmatrix}, \quad \dot{X} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} X + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = (4\ ,\ 3)\, X + 0\, u.$$

Discretize using F.E.: $X_{k+1} = X_k + TAX_k + TBu_k$

$$= \left[\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + 0.1 \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \right] X_k + 0.1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_k$$

$$= \begin{pmatrix} 1 & 0.1 \\ -0.6 & 0.5 \end{pmatrix} X_k + \begin{pmatrix} 0 \\ 0.1 \end{pmatrix} u_k$$

$$y_k = (4\ ,\ 3)\ x_k$$

$$\Rightarrow G_d(z) = C(zI - A)^{-1} B = \frac{0.3z - 0.26}{z^2 - 1.5z + 0.56}$$

NOTE: other ss. realizations are possible but the DT t.f. will be the same

b. F.E transformation: $s = \frac{z-1}{T} \Rightarrow G_d(z) = \frac{3\frac{z-1}{0.1} + 4}{\left(\frac{z-1}{0.1}\right)^2 + 5\frac{z-1}{0.1} + 6}$

$$= \frac{0.3(z-1) + 0.04}{(z-1)^2 + 0.5(z-1) + 0.06} = \frac{0.3z - 0.26}{z^2 - 1.5z + 0.56}$$ (same as before)

c. DT Impulse response: $CA^kB + D$

$$g(k)_{[0:4]} = [D,\ CB,\ CAB,\ CA^2B,\ CA^3B]$$
$$= [0\ ,\ 0.3,\ 0.19\ ,\ 0.117\ ,\ 0.0691]$$ (same as MATLAB's)

(4) 1. Total range: 10V, 2^4 levels $\Rightarrow$ Resolution $\frac{10}{2^4} = 0.625\,V$

2. Max error is 0.625V for a truncating A/D (1 LSB)

0.3125V for a rounding A/D ($\frac{1}{2}$ LSB)

3. Assume that the A/D quantization is a truncation.

Then max error = $10/2^N < 0.001 \Rightarrow 2^N > 10^4 \Rightarrow \underline{\underline{N \geq 14}}$

EEE 481 Homework 1

Problem 1.

Suppose that we measure a signal 0-10V with a 8-bit A/D.

1. What is the resolution?
2. What is the maximum error?
3. How many bits are needed to achieve a maximum error less than 2mV?
4. Assuming that the clock used in the A/D conversion is 5MHz, find the maximum conversion time for a successive approximation converter.

Problem 2.

Consider the system $y(k+2) - \frac{3}{4}y(k+1) + \frac{1}{8}y(k) = x(k+1) - \frac{1}{5}x(k)$, where the multiplications and the addition are quantized to 0.01. Use simulation to assess the mean, worst-case amplitude, and variance of the error due to quantization (compared to non-quantized operation). Apply various inputs x(k), e.g., random, sinusoid, quantized to 0.01. Compare your results with the theoretical bounds computed from the corresponding transfer functions.

1. $\mathrm{mean}(x_n) = G(1)\mathrm{mean}(n)$
2. $\max|x_n(k)| \leq (\sum|g(k)|)\max|n(k)|, (g = Z^{-1}\{G\})$
3. $\mathrm{var}\{x_n(k)\} \leq \left|G\left(e^{j\Omega}\right)\right|_2^2 \mathrm{var}\{n(k)\}$
4. $\mathrm{var}\{x\} \simeq \mathrm{RMS}^2\{x\} \leq \max^2 |G(e^{j\Omega})|\, \mathrm{RMS}^2\{n\}$
5. Use MATLAB's *"linmod"* command to generate the desired transfer functions directly from Simulink models.

Problem 3.

In a data acquisition application we would like to use the Diamond MM board to sample 16 channels with range 0-5V, and transmit the results over the RS-232 serial port.

1. What is the minimum Baud rate required so that the transmission takes less than 0.1sec?
2. What is the maximum error in the A/D conversion?

Problem 4.

A sinusoid with frequency 1Hz is applied to a sampler /ZOH combination. The sampling frequency is 10Hz. List all the frequencies present at the output below 50Hz.

EEE 481 Homework 1 Solutions

Problem 1.

Suppose that we measure a signal 0-10V with a 8-bit A/D.

1. What is the resolution?
2. What is the maximum error?
3. How many bits are needed to achieve a maximum error less than 2mV?
4. Assuming that the clock used in the A/D conversion is 5MHz, find the maximum conversion time for a successive approximation converter.

For min/max values at the ends of the range, the A/D will have 2^n distinct values dividing the interval. Thus, the resolution is $\frac{10-0}{2^n} = 0.039V$. For a truncating A/D the maximum error is the same, 0.39V, and for a rounding A/D the maximum error is ½ LSB = 0.195V. Applying the same formula for different n, we need 13 bits to have error less than 2mV (1.2mV). A successive approximation converter will use roughly 1 clock cycle per bit, so the conversion time is 1.6us.

Problem 2.

Consider the system $y(k+2) - \frac{3}{4}y(k+1) + \frac{1}{8}y(k) = x(k+1) - \frac{1}{5}x(k)$, where the multiplications and the addition are quantized to 0.01. Use simulation to assess the mean, worst-case amplitude, and variance of the error due to quantization (compared to non-quantized operation). Apply various inputs x(k), e.g., random, sinusoid, quantized to 0.01. Compare your results with the theoretical bounds computed from the corresponding transfer functions.

1. $\mathrm{mean}(x_n) = G(1)\mathrm{mean}(n)$
2. $\max|x_n(k)| \le (\sum|g(k)|)\max|n(k)|, (g = Z^{-1}\{G\})$
3. $\mathrm{var}\{x_n(k)\} \le \left|G(e^{j\Omega})\right|_2^2 \mathrm{var}\{n(k)\}$
4. $\mathrm{var}\{x\} \simeq \mathrm{RMS}^2\{x\} \le \max^2 |G(e^{j\Omega})|\, \mathrm{RMS}^2\{n\}$
5. (Optional: Use MATLAB's *"linmod"* command to generate the desired transfer functions directly from Simulink models.)

For a round-off quantization, whose mean is 0 LSB, max(|n|) = ½ LSB = 0.005 and var = 1/3(½ LSB)^2 =8.33e-6. RMS(n) = (var{n})$^{1/2}$ = 0.0029.

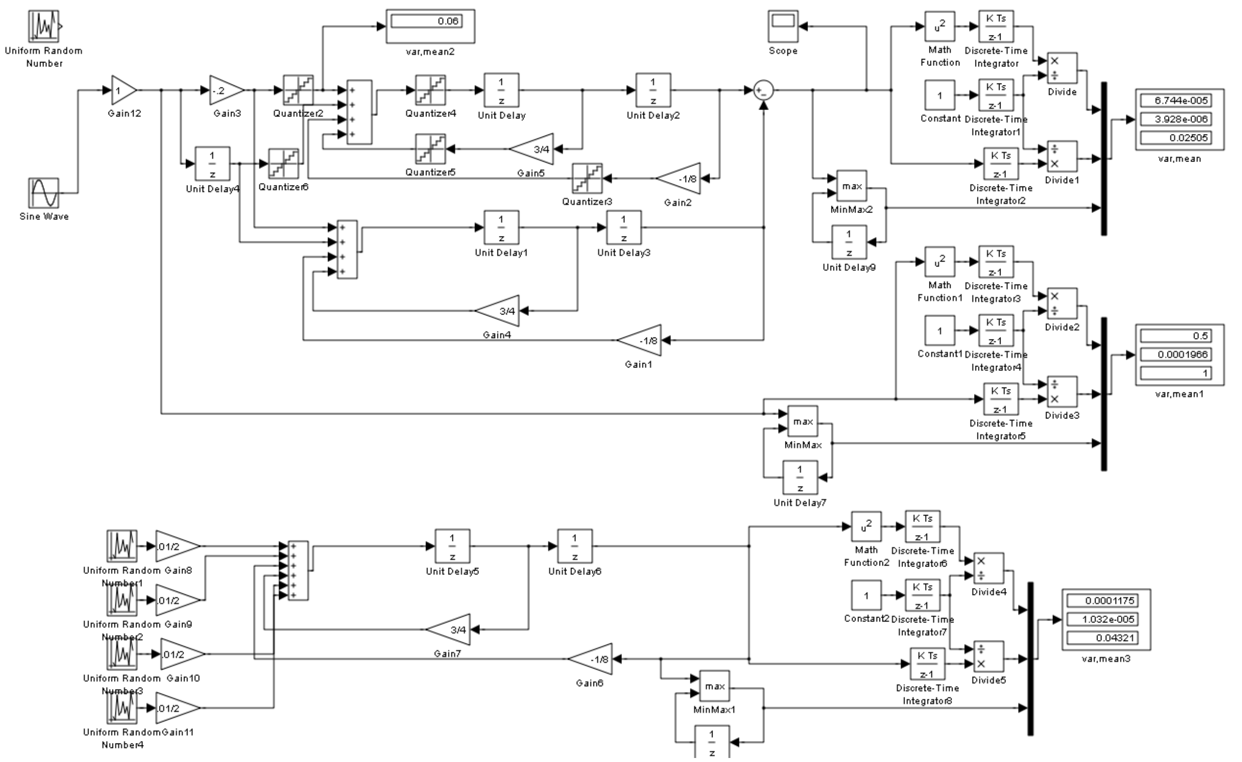
Realizing the transfer function in terms of delays of the output and input (as shown in the figure below) There are 4 quantization blocks, each one contributing ½ LSB uncorrelated noise to the same summation node. The transfer function from each one is $G(z) = \frac{1}{z^2 - \frac{3}{4}z + \frac{1}{8}}$, for which, $G(1) = 2.667, \sum|g(k)| = 2.667, \max\left|G\left(e^{j\Omega}\right)\right| = 2.667, \left|\left|G\right|\right|_2 = 1.352$. Evaluating the above estimates (with x_n denoting the error due to quantization)

1. $\mathrm{mean}(x_n) = G(1)\mathrm{mean}(n)$ =2.667 * 4 * 0 = 0
2. $\max|x_n(k)| \le (\sum|g(k)|)\max|n(k)|, (g = Z^{-1}\{G\}) = 2.667 * 4 * 0.005 = 0.053$
3. $\mathrm{var}\{x_n(k)\} \le \left|G(e^{j\Omega})\right|_2^2 \mathrm{var}\{n(k)\} = 1.352^2 * 4 * 8.33e-6 = 6.09e-5$
4. $\mathrm{var}\{x\} \simeq \mathrm{RMS}^2\{x\} \le (\max|G(e^{j\Omega})|\, \mathrm{RMS}\{n\})^2 \le (2.667 * 4 * 0.0029)^2 = 9.57e-4$

Next, we simulate the quantized system, the ideal system, and the system with the noise model of quantizations and tabulate the results as follows:.

	Rand[-1,1]	Sin(0.1t)	Rand noise model	Estimate
Var	0.056e-3	0.067e-3	0.0615e-3	0.061e-3 [0.95e-3]
Mean	17.5e-6	3.9e-6	78.0e-6	0
Max	0.0302	0.0251	0.0306	0.053

Notice that the stochastic variance estimate (using the 2-norm of G) is close to the observed variance and that the random noise model is fairly representative of the actual errors (for this selection of external inputs). The conservative variance estimates using the RMS deterministic bound (in brackets) is much higher, while the estimate of the maximum amplitude is only conservative by a factor of 2. (This is also because of the specific properties of the system for which sum(|g(k)|) = max|G(e^jw)|.) Also note that for the simulation of the random noise model the random number generators must be initialized with different and appropriate seeds so that they produce uncorrelated outputs.



Problem 3.
In a data acquisition application we would like to use the Diamond MM board to sample 16 channels with range 0-5V, and transmit the results over the RS-232 serial port.

1. What is the minimum Baud rate required so that the transmission takes less than 0.1sec?
2. What is the maximum error in the A/D conversion?

The MM has a 12-bit A/D so, without special compression, it will use 2 Bytes per channel. That is, a total of 32 Bytes per sample time, or 320 Bits (assuming one start, one stop, 8-data; other valid protocols are also acceptable). For the transmission to occur under 0.1sec, the rate should be greater than 3200Baud. The closest standard rate is 4800Baud. Since the MM has 12bit A/D, the maximum error is 1.22mV.

Problem 4.
A sinusoid with frequency 1Hz is applied to a sampler /ZOH combination. The sampling frequency is 10Hz. List all the frequencies present at the output below 50Hz.

From the sampling theorem, replicas of the original signal spectrum will be centered at n x 10Hz frequencies, n = 1, 2, 3, … The ZOH will attenuate all past DC, but its exact zeros are only at n x 10Hz, so it will not zero any of the sinusoid replicas. Thus, the output will contain frequencies
[1, 19, 21, 29, 31, 39, 41, 49]Hz.

EEE 481 Homework 2

Problem 1.
Compute the z-transforms of the following sequences (here u(.) denotes the unit step)

$$u(k-2),\ \sin\left(\frac{\pi}{6}k\right), \qquad \{2-e^{-0.1k}\}u(k), \qquad 0.9^k u(k-1)$$

Problem 2.
Solve the difference equation $y(k+2)-\frac{3}{4}y(k+1)+\frac{1}{8}y(k)= x(k+1)-\frac{1}{5}x(k)$ with the initial conditions y(0)=1, y(-1) = 0 and x(k) = u(k-1).

Problem 3.
Consider the system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned} \quad \text{where} \quad A=\begin{bmatrix} -0.2 & 1 \\ 0 & 0.3 \end{bmatrix} \quad B=\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C=[0.1 \quad 2]$$

1. Determine whether the system is stable or not
2. Determine whether the system is controllable and/or observable
3. Compute its transfer function
4. Compute the first three samples of its unit-step response.

Problem 4.
Write the differential equation describing the motion of a pendulum with input the torque applied at the pivot point and output the angle of the pendulum. Derive the linearized model around the stable and the unstable equilibria and compute the corresponding transfer functions. Assume that the pendulum is a rigid rod of length 0.5m, mass 200g evenly distributed, and its rotation around the pivot point is frictionless.

EEE 481 Homework 2 SOLUTIONS

Problem 1.

Compute the z-transforms of the following sequences (here u(.) denotes the unit step)

$$u(k-2),\ \sin\left(\frac{\pi}{6}k\right),\qquad \{2-e^{-0.1k}\}u(k),\qquad 0.9^k u(k-1)$$

$$Z\{u(k-2)\} = z^{-2}Z\{u(k)\} = \frac{1}{z(z-1)}$$

For a single-sided transform, (k>=0), $Z\left\{\sin\frac{\pi}{6}k\right\} = \frac{z\sin\frac{\pi}{6}}{z^2-2z\cos\frac{\pi}{6}+1}$.

For a double-sided transform, $\sin\frac{\pi}{6}k = \sin\left(\frac{\pi}{6}k\right)u(k) + \sin\left(-\frac{-\pi}{6}k\right)u(-k) - \sin\left(-\frac{-\pi}{6}0\right)\delta(k)$, so

$$Z\left\{\sin\frac{\pi}{6}k\right\} = \left.\frac{z\sin\frac{\pi}{6}}{z^2-2z\cos\frac{\pi}{6}+1}\right|_{ROC:|z|>1} + \left.\frac{z^{-1}\sin\frac{\pi}{6}}{z^{-2}-2z^{-1}\cos\frac{\pi}{6}+1}\right|_{ROC:|z|<1}$$

$$Z\left\{\{2-e^{-0.1k}\}u(k)\right\} = \frac{2z}{(z-1)} - \frac{z}{(z-e^{-0.1})}$$

$$Z\{0.9^k u(k-1)\} = 0.9Z\{0.9^{k-1}u(k-1)\} = 0.9z^{-1}\frac{z}{(z-0.9)} - \frac{0.9}{(z-0.9)}$$

Problem 2.

Solve the difference equation $y(k+2) - \frac{3}{4}y(k+1) + \frac{1}{8}y(k) = x(k+1) - \frac{1}{5}x(k)$ with the initial conditions y(0)=1, y(-1) = 0 and x(k) = u(k-1).

One approach is to define $v(k) = y(k-1)$, so $v(0) = y(-1), v(1) = y(0)$. The ODE, shifted by one, now becomes $v(k+2) - \frac{3}{4}v(k+1) + \frac{1}{8}v(k) = x(k) - \frac{1}{5}x(k-1)$. Taking transforms and applying the initial condition property, we get

$$z^2V(z) - z^2v(0) - zv(1) - \frac{3}{4}zV(z) + \frac{3}{4}zv(0) + \frac{1}{8}V(z) = X(z) - \frac{1}{5}z^{-1}X(z)$$

Substituting the IC and X(z),

$$\left[z^2 - \frac{3}{4}z + \frac{1}{8}\right]V(z) = z + \left(1 - \frac{1}{5z}\right)\frac{1}{z-1}$$

After PFE,

$$V(z) = \frac{2}{z-0.5} + \frac{-1}{z-0.25} + \frac{2.133z^{-1}}{z-1} + \frac{-2.4z^{-1}}{z-0.5} + \frac{0.267z^{-1}}{z-0.25}$$

Hence,

$$v(k) = 2(0.5)^{k-1}u(k-1) - (0.25)^{k-1}u(k-1) + 2.133u(k-2) - 2.4(0.5)^{k-2}u(k-2) + 0.267(0.25)^{k-2}u(k-2)$$

And,

$$y(k) = 2(0.5)^k u(k) - (0.25)^k u(k) + 2.133u(k-1) - 2.4(0.5)^{k-1}u(k-1) + 0.267(0.25)^{k-1}u(k-1)$$

(Notice: depending on the approach, one may obtain different but equivalent expressions, e.g.,
$2(0.5)^k u(k) - (0.25)^k u(k) = 1\delta(k) + 2(0.5)(0.5)^{k-1}u(k-1) - (0.25)(0.25)^{k-1}u(k-1) \Rightarrow$
$y(k) = \delta(k) - 1.4(0.5)^{k-1}u(k-1) + 0.017(0.25)^{k-1}u(k-1) + 2.133u(k-2)$, etc.)

Problem 3.
Consider the system

$$x_{k+1} = Ax_k + Bu_k \quad \text{where} \quad A = \begin{bmatrix} -0.2 & 1 \\ 0 & 0.3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$y_k = Cx_k \qquad\qquad C = [0.1 \quad 2]$$

1. Determine whether the system is stable or not
2. Determine whether the system is controllable and/or observable
3. Compute its transfer function
4. Compute the first three samples of its unit-step response.

1. The eigenvalues of A are -0.2 and 0.3, they are inside the unit circle, hence the system is stable.
2. The controllability matrix [B,AB] has rank 2, so the system is completely controllable. The observability matrix [C;CA] has rank 2, so the system is completely observable.
3. The transfer function is $C(zI - A)^{-1}B + D = \frac{2z+0.5}{z^2-0.1z-0.06}$
4. We compute the recursion for the states, starting with x(0)=0 and u(k)=1 for k>=0. Then,
y(0) = 0
y(1) = 2
y(2) = 2.7
y(3) = 2.89
etc.

Problem 4.
Write the differential equation describing the motion of a pendulum with input the torque applied at the pivot point and output the angle of the pendulum. Derive the linearized model around the stable and the unstable equilibria and compute the corresponding transfer functions. Assume that the pendulum is a rigid rod of length 0.5m, mass 200g evenly distributed, and its rotation around the pivot point is frictionless.

Newton's law yields, $J\frac{d^2\theta}{dt^2} = -mg\frac{L}{2}\sin\theta + u$, where $J = \frac{1}{3}mL^2$. Hence, substituting the pendulum parameters, $\frac{d^2\theta}{dt^2} = -30\sin\theta + 60u$.

The linearized system around the stable equilibrium has $\sin\theta \simeq \cos 0\,\theta_L, \theta_0 = \theta_0' = u_0 = 0$,

$$\frac{d^2\theta_L}{dt^2} = -30\theta_L + 60u_L, \qquad \frac{\theta_L(s)}{u_L(s)} = \frac{60}{s^2+30}$$

The linearized system around the unstable equilibrium has $\sin\theta \simeq \cos\pi\,\theta_L, \theta_0 = \pi, \theta_0' = u_0 = 0$,

$$\frac{d^2\theta_L}{dt^2} = 30\theta_L + 60u_L, \qquad \frac{\theta_L(s)}{u_L(s)} = \frac{60}{s^2-30}$$

EEE 481 **Homework 3**

Problem 1.

Consider the continuous time system with transfer function $G(s) = \frac{3s+2}{s^2+5s+4}$.

1. Realize G(s) in state-space and use Forward Euler to compute its discretization, using sampling time T = 0.1. Find the transfer function of the corresponding discrete-time system.
2. Use Forward Euler directly on the transfer function G(s) and compute the corresponding discrete-time transfer function. Realize the discrete-time system in state-space.
3. Compute the first five terms of the discrete-time system impulse response using state-space formulae. Compare with the result of MATLAB's *impulse(.)* function.

Problem 2.

The first-principles model of a temperature control system is $\dot{Y} = -0.2(Y-273)+Q$, where Y is the Temperature (Kelvin) and Q is the supplied heat (Watts).

1. Use the Forward Euler approximation of derivative $\dot{Y}(t_k) \cong \frac{Y(t_{k+1})-Y(t_k)}{T_s}$ to write a corresponding discrete time state-space model for a sampling time of 2sec.
2. What is the discrete-time transfer function of the system?
3. What are the limitations (if any) of this discretization method.

Problem 3.

An analog filter with the transfer function $\frac{1}{(2s+1)(0.01s+1)}$ is to be replaced by a computer. Determine an appropriate sampling time and the transfer function of the discretized filter. You may use any discretization method you like but you should justify all choices.

EEE 481 Homework 3

Problem 1.

Consider the continuous time system with transfer function $G(s) = \frac{3s+2}{s^2+5s+4}$.

1. Realize G(s) in state-space and use Forward Euler to compute its discretization, using sampling time T = 0.1. Find the transfer function of the corresponding discrete-time system.
2. Use Forward Euler directly on the transfer function G(s) and compute the corresponding discrete-time transfer function. Realize the discrete-time system in state-space.
3. Compute the first five terms of the discrete-time system impulse response using state-space formulae. Compare with the result of MATLAB's *impulse(.)* function.

1. $\dot{x} = Ax + Bu, y = Cx + Du,\ [A, B, C, D] = \left\{\begin{bmatrix} -5 & -2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, [1.5\ \ 5], [0]\right\}$, is one possible realization. The FE discretization is found by $\dot{x}(T) \simeq \frac{x_{k+1}-x_k}{T} \Rightarrow x_{k+1} = (I + TA)x_k + TBu_k$, $y_k = Cx_k + Du_k$. The transfer function for the discrete time system becomes

$$G_d(z) = C(zI - [I + TA])^{-1}[TB] + D$$

This transfer function can be computed by hand, or by the following MATLAB commands

```
>> G=tf([3,2],[1,5,4])
>> Gs=ss(G)
>> T=.1;Gd=ss(eye(size(Gs.a))+Gs.a*T,Gs.b*T,Gs.c,Gs.d,T);tf(Gd)

ans =

   0.3 z - 0.28
 ------------------
 z^2 - 1.5 z + 0.54

Sample time: 0.1 seconds
```

Notice that the transfer function does not depend on the choice of realization of the continuous transfer function. The above procedure can therefore be used to find the FE discretization of a continuous time system.

2. Performing the substitution $s = \frac{z-1}{T}$, we find exactly the same discrete transfer function as in Part 1. A state space realization is

$x_{k+1} = Ax_k + Bu_k,\ y_k = Cx_k + Du_k,\ [A, B, C, D] = \left\{\begin{bmatrix} 1.5 & -0.54 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [0.3\ \ 0.28], [0]\right\}$,

which does not need to be (and is not) the same as the one in Part 1.

3. We can easily compute the recursion $x_{k+1} = Ax_k + Bu_k,\ y_k = Cx_k + Du_k$, with IC = 0 and $u_k = 1\ for\ k = 0, 0\ oth.$:

```
>> x = [0;0];
>> y=c*x, x=a*x+b;
>> y=c*x, x=a*x;
>> y=c*x, x=a*x;
Etc.
```

We find the values for y: 0, 0.3000, 0.1700, 0.0930, 0.0477, 0.0213

We also find the same values with h = impulse(Gd).

Problem 2.

The first-principles model of a temperature control system is $\dot{Y} = -0.2(Y - 273) + Q$, where Y is the Temperature (Kelvin) and Q is the supplied heat (Watts).

1. Use the Forward Euler approximation of derivative $\dot{Y}(t_k) \cong \dfrac{Y(t_{k+1}) - Y(t_k)}{T_s}$ to write a corresponding discrete time state-space model for a sampling time of 2sec.
2. What is the discrete-time transfer function of the system?
3. What are the limitations (if any) of this discretization method.

1. $Y_{k+1} - Y_k = -0.4Y_k + 109.2 + 2Q_k$
2. From Q to Y, the discrete transfer function is $\frac{Y(z)}{Q(z)} = \frac{2}{z-0.6}$. (273 can be viewed as an external input, or the output can be interpreted as the incremental output over the equilibrium solution Y = 273 for Q = 0.)
3. The stability constraint for the discrete model is $|1 - 0.2T_s| < 1 \Rightarrow T_s < 10$. Of course, for a sensible approximation, the sampling time should be much less that this bound. E.g., one-half the value will produce a discretized system with pole at the origin, i.e., the entire dynamic response is modeled by a single delay.

Problem 3.

An analog filter with the transfer function $\dfrac{1}{(2s+1)(0.01s+1)}$ is to be replaced by a computer. Determine an appropriate sampling time and the transfer function of the discretized filter. You may use any discretization method you like but you should justify all choices.

A reasonable choice for the sampling time would be related to the system bandwidth (0.5rad/s). One may choose different rules of thumb.

- 6 samples/rise time: tr = 2/BW = 4s; T = 4/6 = 0.67s or f = 1.5Hz. (Measuring tr from a step response simulation we find 4.4s which is reasonably close).
- Nyquist = 10 x BW = 5rad/s = 0.8Hz => f = 1.6Hz, T = 0.625s. (This is similar to the above since tr = 2/BW (BW in rad/s) => T = 1/(3BW) = 1/(6piBW) (BW in Hz) => f = 19BW (BW in Hz).)
- ZOH adds 6deg phase lag at BW (a feedback-related spec), wT/2 = 0.1 => T = (2/BW)0.1 = 0.4s (9 deg phase lag yields the previous T = 0.63s)

Since we are trying to replace an analog filter and have a discretization with similar filtering properties, a Tustin discretization is the more reasonable choice. Thus, for T = 0.625s (not a unique choice), the discretized transfer function is

$$H_d(z) = \frac{0.1309\,\mathrm{z}^2 + 0.2619\,\mathrm{z} + 0.1309}{\mathrm{z}^2 + 0.2083\,\mathrm{z} - 0.6845}$$

One potential drawback of this solution is that it is bi-proper (y_k requires u_k)

However, for a Forward Euler discretization, the sample time is constrained by the fastest sampling constant (0.01). Here, choosing T = 0.01 will transform this term to $0.01\ \frac{(z-1)}{0.01} + 1 = z$, i.e., the entire response is approximated by a single delay. Since the system response is dominated by the slower mode (2s+1), such an approximation is acceptable (assuming of course that such an oversampling is possible). For this case,

$$H_d(z) = \frac{0.005}{\mathrm{z}(\mathrm{z} - 0.995)}$$

EEE 481 Homework 4

Problem 1.

Consider the following system with transfer function $P(s) = \dfrac{1}{(s+0.1)(s+1)}$.

1. Design a PID so that the closed loop crossover is at 7rad/s and the phase margin is 50^{o}.
2. Select a method and the sampling frequency and discretize the PID.

Problem 2.

1. Ziegler-Nichols Tuning: Apply the two Z-N methods from the notes to tune a PID for the plants:
$$P_1(s) = \frac{-0.1s + 2}{s^2 + 4s + 2} \quad P_2(s) = \frac{20(-0.2s + 1)}{s^2 + 10s + 20}$$
2. Compare the results with a PID designed for a gain crossover frequency of open-loop bandwidth and 45deg. phase margin.

*Hint: Define P as a transfer function object and use step(P) to get an estimate of R,L for the first Z-N tuning. Then iterate k on step(feedback(k*P,1)) until the system is marginally stable (slowly increasing or slowly decreasing response). Then estimate Ku,Pu for the second Z-N tuning. Define the compensators and compare step responses and bode plots for the transfer functions command-to-output and input disturbance-to-output*

Problem 3.

1. Design a PID controller to achieve a bandwidth of 1Hz, 50deg phase margin, and to be discretized with a sampling frequency of 10Hz for the system with transfer function
$$P_1(s) = \frac{-0.1s + 1}{s^2 + 4s + 2}$$
2. Compare the results with a design in discrete time directly, where the plant is discretized and the parameters of a discrete-time PID are calculated to achieve the same specifications.

EEE 481 **Homework 4** **SOLUTIONS**

Problem 1.

Consider the following system with transfer function $P(s) = \dfrac{1}{(s+0.1)(s+1)}$.

1. Design a PID so that the crossover is at 7rad/s and the phase margin is 50^o.
2. Select a method and the sampling frequency and discretize the PID.

For a discrete design we should first select the sample time to contribute, say, -3deg phase at crossover, i.e., w_cT /2= 0.105/2 or T = 0.015 sec. The phase of P alone at 7 rad/s is -171 deg, so we need a PID to control it. We define:

$$C(s) = \frac{K(s+a)^2}{s(\tau s+1)}$$

To achieve 50 degrees phase margin with the discrete controller, we should compute the PID zeros to provide 50+3 deg phase margin. Here, however, the problem asks for 50 degrees PM: $2\tan^{-1}\frac{7}{a} - 90^o - \tan^{-1} 7\tau = -130^o \Rightarrow 2\tan^{-1}\frac{7}{a} = 137$, for $\tau = T$. Then, we compute $a = \frac{7}{\tan\frac{137}{2}} = \frac{7}{2.54} = 2.76$. Substituting back to the gain equation $|P(j7)C(j7)| = 1 \Rightarrow K = 6.15$. Computing the margins for PC we verify the design.

The sampling frequency is now 1/T = 66.7 Hz and the preferred method of discretization of the PID is Tustin, for which we expect a phase margin of 47 deg., since we did not pre-compensate for the ZOH. The controller has the transfer function

$$C_d(z) = \frac{284.9\,z^2 - 546.8\,z + 262.3}{(z-1)(z-0.333)}$$

If we evaluate its margins, it provides a 46.9 degree PM, very close to the expected value. The step and frequency responses are also very close to the continuous time versions.

Problem 2.

1. Ziegler-Nichols Tuning: Apply the two Z-N methods from the notes to tune a PID for the plants:

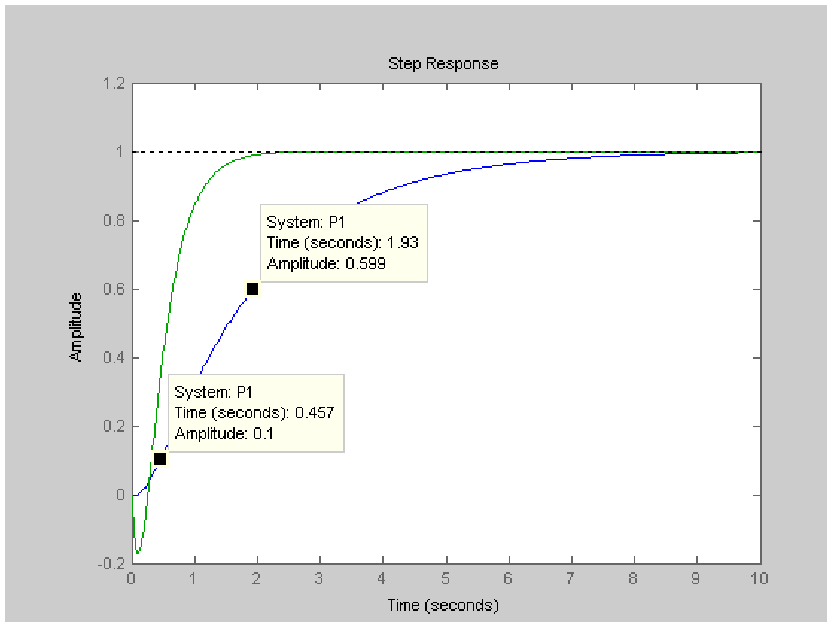
$$P_1(s) = \frac{-0.1s+2}{s^2+4s+2} \quad P_2(s) = \frac{20(-0.2s+1)}{s^2+10s+20}$$

2. Compare the results with a PID designed for a gain crossover frequency of open-loop bandwidth and 45deg. phase margin.

*Hint: Define P as a transfer function object and use step(P) to get an estimate of R,L for the first Z-N tuning. Then iterate k on step(feedback(k*P,1)) until the system is marginally stable (slowly increasing or slowly decreasing response). Then estimate Ku,Pu for the second Z-N tuning. Define the compensators and compare step responses and bode plots for the transfer functions command-to-output and input disturbance-to-output*

We compute the approximate slopes from the step responses as
R1 = 0.34, L1 = 0.21, R2=1.47, L2= 0.26.
The corresponding controllers are

$$C_1(s) = \frac{1.47\,s^2 + 16.8\,s + 40.}{s}$$
$$C_2(s) = \frac{0.34\,s^2 + 3.18\,s + 6.2}{s}$$



For the second method, we try closing the loop with different gains, until oscillatory response is observed. For the first system we find Ku1 = 40, Pu1 = 0.70 and for the second Ku2 = 2.5, Pu1 = 0.74.
The corresponding controllers are

$$C_1(s) = \frac{2.09\,s^2 + 24\,s + 68.8}{s}$$
$$C_2(s) = \frac{0.139\,s^2 + 1.5\,s + 4.05}{s}$$

Note that while these gain values happened to produce an exact oscillatory response (due to the round numbers in the system transfer functions), this does not need to be the case in general; for practical applications, sufficient approximation can be obtained by gains that produce decaying oscillations with low damping.

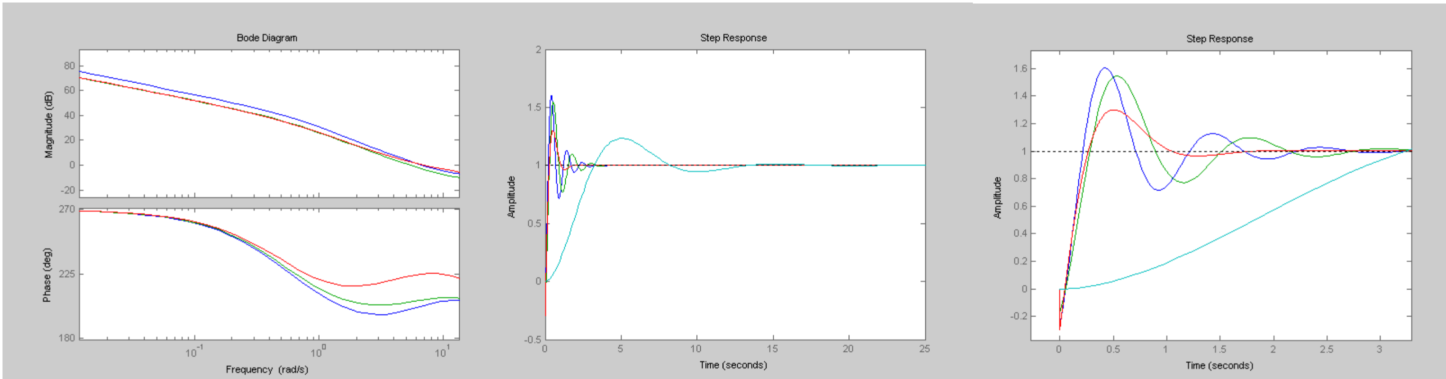
Finally, we design a controller for crossovers at the open-loop bandwidth: The first system has BW = 0.577rad/s and the second has BW = 3.2rad/s. Performing the design, we find

$$C_1(s) = \frac{0.266\,s + 0.81}{s}; \quad C_2(s) = \frac{0.112\,s^2 + 1.23\,s + 3.36}{s}$$

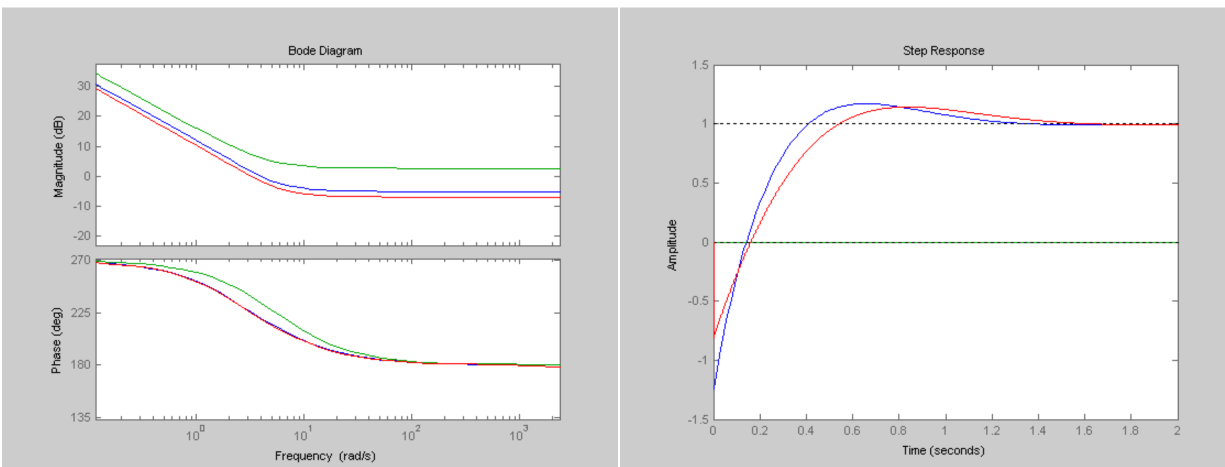
It turns out that this controller is too slow and not comparable with the Z-N. Instead, we can match the Z-N bandwidth with a crossover at 12x BW:

$$C_1(s) = \frac{2.83\,s^2 + 21.2\,s + 40}{s}$$

The step responses with these controllers are shown in the following figures. (Bode plot of loop-tf, step responses. Fbk-ZN: blue, OL-ZN: green, PM-tuned: red -cyan is the slow one). We see that both ZN yield good and similar responses, even though the damping is lower than the 45deg. phase margin controller.



For the second system the Fbk-ZN method and yields similar results to the Phase margin tuning but the OL-ZN does not produce a stabilizing controller. The plant does not attenuate the high frequencies enough and the OL-ZN method yields a very optimistic estimate of the controller gain that fails drop below unity.



Problem 3.

1. Design a PID controller to achieve a bandwidth of 1Hz, 50deg phase margin, and to be discretized with a sampling frequency of 10Hz for the system with transfer function
$$P(s) = \frac{-0.1s + 1}{s^2 + 4s + 2}$$
2. Compare the results with a design in discrete time directly, where the plant is discretized and the parameters of a discrete-time PID are calculated to achieve the same specifications.

We design continuous controller with an additional PM corresponding to the ZOH half-sample delay Tw_c/2 = 12 deg. We also select the sample time as 0.05s, anticipating the Tustin transformation, to yield the PID poles at 0 and 1. The angle required by the two PID zeros is 138deg and the final controller is
$$C(s) = \frac{4.52\, s^2 + 14.0\, s + 10.9}{0.05\, s^2 + s}$$
Using the Tustin transformation, we obtain the discrete-time controller
$$C_d(z) = \frac{52.5\, z^2 - 89.9\, z + 38.5}{z^2 - z}$$
This controller yields a Phase Margin of 50 deg, at ~4.1 rad/s, as requested. Note that this tuning is near the limits of what can be achieved with PIDs and sampling rate becomes important since the differentiator pole and the ZOH contribute a substantial -24 deg phase lag at crossover, while the crossover is less than one decade below Nyquist frequency.

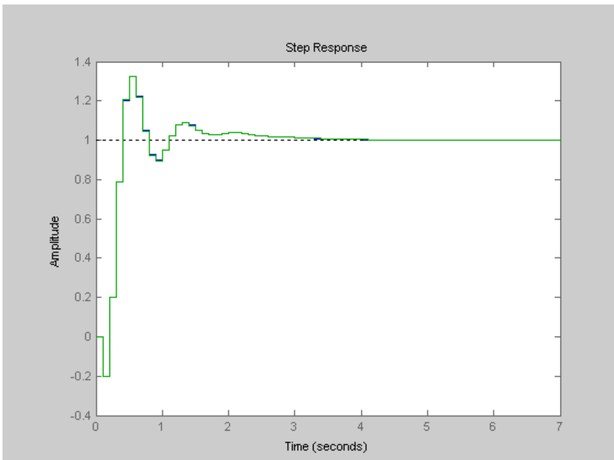
Next, we consider an entirely discrete time design. We compute the ZOH-equivalent of the plant
$$P_d(z) = \frac{-0.00383\, z + 0.0121}{z^2 - 1.65\, z + 0.670}$$
And consider the discrete PID
$$C_d(z) = \frac{K\,(z-a)^2}{z^2 - z}$$
(For consistency, we maintain the same PID poles in the two cases.)
We compute the angle of the plant and the PID poles at crossover:
>> [m,p]=bode(P1d*dp,4.2)

p - 360 =

 -2.9425e+02

For 50 deg phase margin, this requires an angle contribution from each zero of 82.1 deg.
$$a\text{: atan}\frac{\sin\Omega}{\cos\Omega\ - a} = 82.12^o \Rightarrow a = \frac{7.23\cos 0.42 - \sin 0.42}{7.23} = 0.857$$
We then compute the gain K so that the crossover is at 4.2 rad/s (0.42 rad/sample), K = 1/ 1.8999e-02. The final controller is
$$C_d(z) = \frac{52.6\, z^2 - 90.2\, z + 38.7}{z^2 - z}$$

Obviously, both methods yield very similar controllers and responses. (Any differences are expected to appear much closer to the Nyquist frequency.)



EEE 481 Homework 5

Problem 1.

1. Design a PID controller to achieve a bandwidth of 1Hz, 50deg phase margin, and to be discretized with a sampling frequency of 10Hz for the system with transfer function
$$P_1(s) = \frac{-0.1s + 1}{s^2 + 4s + 2}$$
2. An additive disturbance enters the plant output with transfer function $P_2(s) = \frac{1}{0.1s+1}$. Design a feedforward component for the PID controller, also discretized at 10Hz, to reduce the effect of the disturbance on the output.

Problem 2.

The read arm on a computer disk drive has transfer function
$$\mathrm{H(s)} = \frac{1000}{\mathrm{s}^2}$$

1. Design an analog PID controller to achieve a bandwidth of approx. 100Hz with 45deg. phase margin.
2. Design a discrete PID for the same bandwidth and phase margin, with a sampling frequency 1kHz and simulate the closed loop step response.
3. What is the maximum bandwidth that can be achieved with a PID having 45deg phase margin and 1kHz sampling?
4. Design a prefilter to achieve overshoot to step reference changes under 5%.

Hint: You need a complete PID for this problem (2-zeros). Use a filter for the pseudo-differentiator with T = 0.001, consistent with the 1ms sampling time.

Problem 3.

Design a PID controller for the flexible inverted pendulum with transfer function
$$\frac{\{1.478\}}{\{\mathrm{s}^2 + 0.0635\mathrm{s} - 19.54\}} + \frac{\{0.000332\,\mathrm{s}^2 + 0.3785\,\mathrm{s} + 177.5\}}{\{\mathrm{s}^2 + 15.52\,\mathrm{s} + 64750\}}$$
For this problem, the PID should be augmented by a low-pass filter to increase roll-off beyond bandwidth and avoid the excessive excitation of the flexible modes. The sampling frequency is 1000Hz and the choice of closed-loop bandwidth is left as a design parameter. Use a 3rd order low-pass filter, with bandwidth roughly at 2x or 3x of the crossover frequency. In your design, include a prefilter to maintain overshoot to step reference changes under 5%. Verify the stability of your controller with simulations.

EEE 481, Homework 5, Solutions

Problem 1.

1. Design a PID controller to achieve a bandwidth of 1Hz, 50deg phase margin, and to be discretized with a sampling frequency of 10Hz for the system with transfer function
$$P_1(s) = \frac{-0.1s + 1}{s^2 + 4s + 2}$$
2. An additive disturbance enters the plant output with transfer function $P_2(s) = \frac{1}{0.1s+1}$. Design a feedforward component for the PID controller, also discretized at 10Hz, to reduce the effect of the disturbance on the output.

1. We allow for 12deg extra phase margin to deal with the ZOH. The continuous PID is
cpid =

```
  4.521 s^2 + 14.01 s + 10.86
  ---------------------------
        0.05 s^2 + s
```

The discretized PID (Tustin) is

dpid =

```
  52.49 z^2 - 89.89 z + 38.48
  ---------------------------
           z^2 - z
```

Sample time: 0.1 seconds

2. For the prefilter, we can work either in continuous time (but that requires the redesign of the PID without the ZOH to obtain the equivalent continuous time closed loop system) or in discrete time (but here the factorization function operate in continuous time so a Tustin transform is necessary). Other than that, we follow the stable projection algorithm described in the notes.

As alternatives, we note the approximation of the Plant-inverse by the inverse of its outer part (i.e., the invertible part, where the RHP zeros are replaced by their mirror images (for C-T systems, or their inverses for D-T systems). This approximation has the correct magnitude but its phase is approximately correct only for low frequencies. Similarly, one can also use the DC of the plant as an approximation so the FF is simply $(\mathrm{P}(0)^{-1}) * \mathrm{Q}$.

A code implementing the solution is given below.

```
+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
%EEE 481, HW 5, Problem 1

a=-.1
P=tf([a 1],[1 4 2])
Q=tf(1,[.1 1])
[pid1,cpid]=pidpmtune(6.28,P,.05,50+12)
dpid=c2d(cpid,.1,'tustin')
Pd=c2d(P,.1)
Qd=c2d(Q,.1)
% step(fbk(Pd*dpid,1));pause
```

```
% bode(fbk(Pd*dpid,1));pause
% margin(Pd*dpid);pause

Sd=feedback(1,Pd*dpid); SPd=feedback(Pd,dpid); SQd=Sd*Qd;
W=1,r=1e-6,W=c2d(tf([.1 1],[1 1e-4]),.1,'tustin');
Gd=[W*SPd;r]; WTd=([W*SQd;0]);
[SPi,SPip,SPo]=iofr(ss(d2c(Gd,'tustin'))); Stil=inv([SPi,SPip]);
R=minreal(Stil*d2c(WTd,'tustin'));
X2=stabproj(R-R.d)+R.d;  H2o=minreal(inv(SPo)*[1 0]*X2);
eig(H2o)
cut=[];
while isempty(cut), cut=input('cut '),end
[H2s,H2f]=slowfast(H2o-H2o.d,cut);H2f=H2f+H2o.d;
Hd=c2d(H2f,.1,'Tustin');      % H2 optimal design

[Hi,Hip,Ho]=iofr(ss(d2c(Pd,'tustin')));Hd_alt=c2d(inv(Ho)*Q,.1,'Tustin');
                              % Outer Approximation
Hd_dc=inv(dcgain(Pd))*Qd;     % DC-gain Approximation

step(Sd*Qd,SPd*Hd,SPd*Hd_alt,SPd*Hd_dc);pause
sigma(Sd*Qd,Sd*Qd-SPd*Hd,Sd*Qd-SPd*Hd_alt,Sd*Qd-SPd*Hd_dc)
disp('Error system Norms:')
disp('    No-FF,  H2-optimal, Outer app,  DC-gain')
disp([norm(W*(Sd*Qd)),norm(W*(Sd*Qd-SPd*Hd)), ...
      norm(W*(Sd*Qd-SPd*Hd_alt)),norm(W*(Sd*Qd-SPd*Hd_dc))])

++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

function [pid,cpid]=pidpmtune(bw,g,tau,pm,n)
% [pid,cpid]=pidpmtune(bw,g,tau,pm)
% bw = bandwidth
% g = system
% tau = derivative TC
% pm = phase margin
% n = PI/PID order (forced)

if nargin<3,tau=.01; end
if isempty(tau),tau=.01;end
if nargin<4;pm=50;end
if isempty(pm);pm=50;end
if nargin<5;n=[];end

gc=bw/1.5;

cpid=tf(1,[1,0]);
[m,p]=bode(g*cpid,gc);
p=mod(p,360);
if p>0;p=p-360;end
th=(-180-p+pm)
if isempty(n)
    if abs(th)>75; n=2;else;n=1;end
end

if n==1;cpid=tf(1,[1 0]);else;cpid=tf(1,[tau,1,0]);end
[m,p]=bode(g*cpid,gc);
p=mod(p,360);
if p>0;p=p-360;end
th=(-180-p+pm);
```

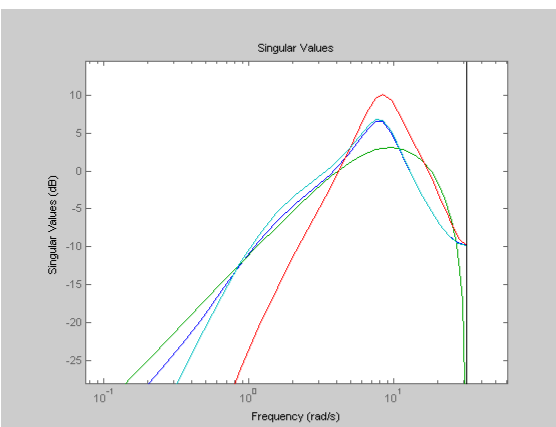
```
a=gc/tan(abs(th)/n*pi/180)
if n==2;cpid=tf([1 2*a a*a],[tau,1,0]);else;cpid=tf([1 a],[1,0]);end
[m,p]=bode(g*cpid,gc);
cpid=cpid*(1/m);

[nu,de]=tfdata(cpid,'v');

if length(nu)<3;nu=[0,nu];end
pid=[nu(2)-nu(3)*tau,nu(3),nu(1)-tau*(nu(2)-nu(3)*tau)];
```

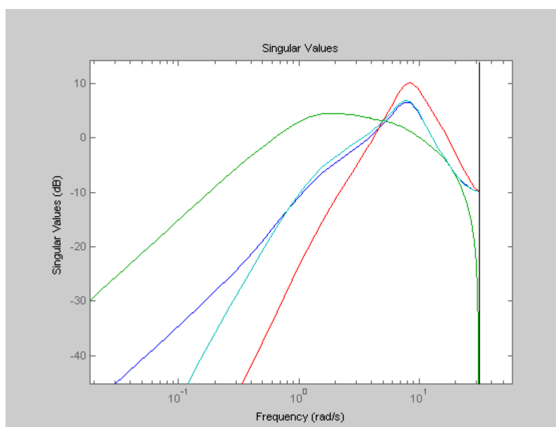
++

As a metric of the success of the design, we compute the error system norms, which describe the amplification of the variance of the disturbance signal. We observe that the Outer approximation produces the lowest error at low frequencies but the highest variance amplification. Also notice that the proximity of the RHP zero to the bandwidth results in a small performance improvement over the no-feedforward case.

```
Error system Norms:
    No-FF,  H2-optimal, Outer appr,  DC-gain
    0.1792    0.1643    0.2190    0.1883
```

(The Ad hoc solution of the outer approximation is actually worse than no feedforward, in terms of variance to white noise disturbance!)





The results are qualitatively similar for un-weighted approximations (right figure, W=1; blue = no FF, green, H2-optimal FF, red: Outer Approximation FF, cyan: DC Gain Approximation) but the step responses and error frequency responses are not very appealing. (The optimal solutions are sometimes unexpected, especially if one does not carefully define what is the objective.)

```
Error system Norms:
    No-FF,  H2-optimal, Outer app,  DC-gain
    0.9880    0.8818    1.4504    1.0196
```

For the record, the transfer function of the H2-optimal prefilter with the low-pass weighting function is (but further reduction is likely to be feasible):

```
    2.05 z^5 + 15.45 z^4 - 68.59 z^3 + 91.54 z^2 - 49.82 z + 9.059
  --------------------------------------------------------------------
  z^5 - 1.019 z^4 + 0.3452 z^3 - 0.03893 z^2 + 4.708e-07 z - 5.219e-08
```

```
The prefilter with no weighting is:
```

```
   -36.75 z^4 + 52.35 z^3 - 47.17 z^2 + 31.11 z - 6.915
  -------------------------------------------------------
  z^4 - 0.6853 z^3 + 0.1168 z^2 - 8.636e-09 z + 1.174e-09
```

```
Sample time: 0.1 seconds

The DC-approximation, in both cases, is

    1.264
  ----------
  z - 0.3679
```

Problem 2.

The read arm on a computer disk drive has transfer function

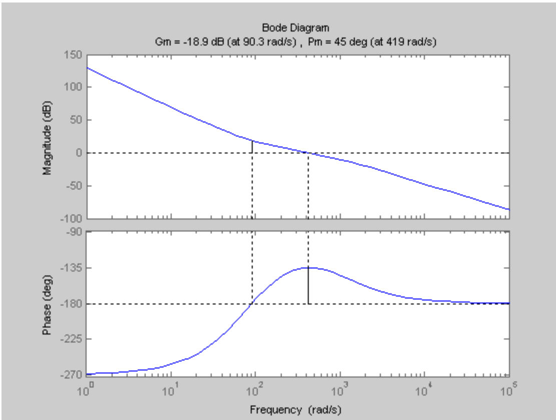
$$H(s) = \frac{1000}{s^2}$$

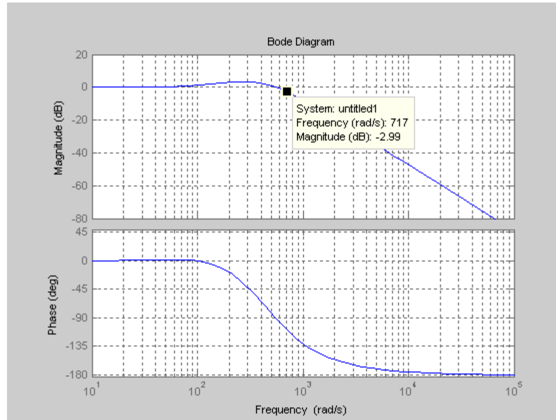
1. Design an analog PID controller to achieve a bandwidth of approx. 100Hz with 45deg. phase margin.
2. Design a discrete PID for the same bandwidth and phase margin, with a sampling frequency 1kHz and simulate the closed loop step response.
3. What is the maximum bandwidth that can be achieved with a PID having 45deg phase margin and 1kHz sampling?
4. Design a prefilter to achieve overshoot to step reference changes under 5%.

Hint: You need a complete PID for this problem (2-zeros). Use a filter for the pseudo-differentiator with T = 0.001, consistent with the 1ms sampling time.

The continuous-time design is following the standard procedure, with 45 deg. phase margin and 628rad/s as intended closed loop bandwidth. The resulting closed loop has the correct phase margin and slightly larger bandwidth (718 rad/s, no iteration is necessary here).

```
>> [pid1,cpid]=pidpmtune(628,P,.001,45)

cpid =

  436.9 s^2 + 7.205e04 s + 2.97e06
  --------------------------------
          0.001 s^2 + s
```





Next, the discrete-time design will be computed by discretizing the continuous controller but with an adjusted phase margin to compensate for the ZOH. This angle is $\frac{wT}{2} = \frac{628}{1.5} \times \frac{0.001}{2}(rad) = 12^o$

>> [pid2,cpidd]=pidpmtune(628,P,.001,45+12)

cpidd =

```
  450.2 s^2 + 3.392e04 s + 6.389e05
  ---------------------------------
           0.001 s^2 + s
```

And the discrete pid is found as its Tustin-equivalent:

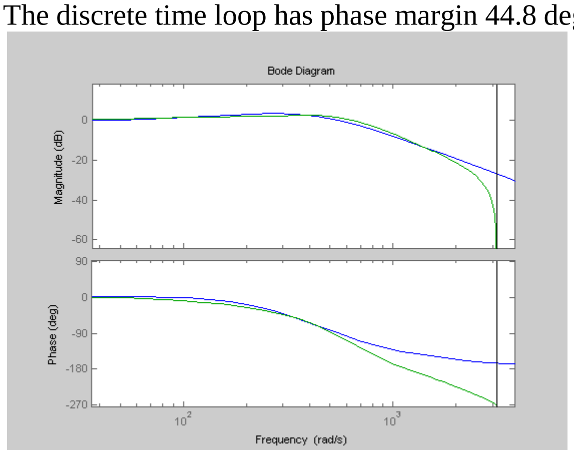
>> dpid=c2d(cpidd,.001,'tustin')

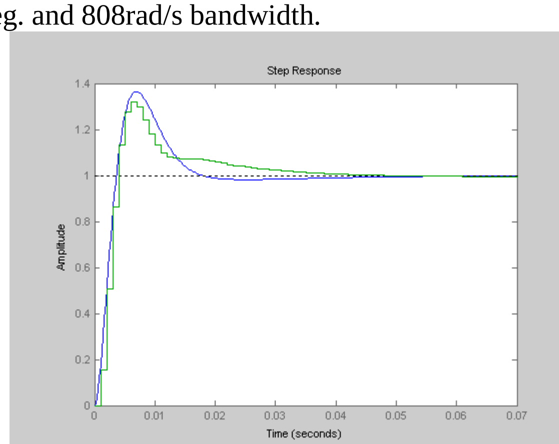
dpid =

```
  3.116e05 z^2 - 6.001e05 z + 2.89e05
  -----------------------------------
        z^2 - 1.333 z + 0.3333
```

Sample time: 0.001 seconds

The discrete time loop has phase margin 44.8 deg. and 808rad/s bandwidth.





The frequency and step responses of the closed-loop systems are fairly close to each other.

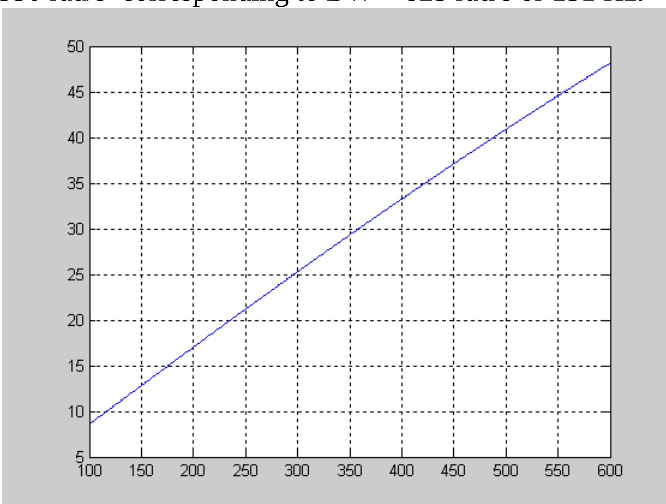
The phase margin equation for this system is

$$-180-90-\operatorname{atan}\frac{w}{1000}-\frac{w}{2000}\times\frac{180}{\pi}+2\operatorname{atan}\frac{w}{a}=-180+45$$

Where a is the PID zero; this yields the constraint

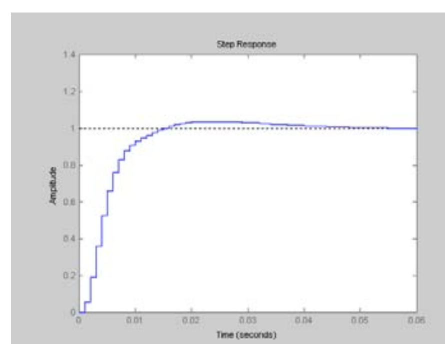
$$2\operatorname{atan}\frac{w}{a}=135+\operatorname{atan}\frac{w}{1000}+\frac{w}{2000}\times\frac{180}{\pi}\leq 180 \Rightarrow \operatorname{atan}\frac{w}{1000}+\frac{w}{2000}\times\frac{180}{\pi}\leq 45$$

We plot the left-hand side as a function of w to find that the maximum possible crossover frequency is 550 rad/s corresponding to BW ~ 825 rad/s or 131 Hz.



Of course, for a reasonable integral action, the PID zero should not contribute more than 75-80 degrees, bringing the practically feasible crossover below 300 rad/s or BW ~ 450 rad/s (= 71 Hz). Our design calls for a higher bandwidth, and the effect of the slow zero can be seen in the slow residual convergence of the step response. (The C-T design has an extra 12 degrees of room in the angle equation and it does not exhibit this problem.)

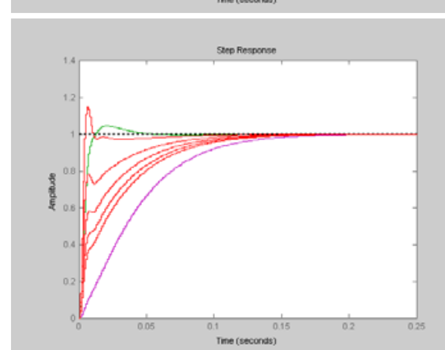
4. For a prefilter, we can use the general procedure of an additive signal at the plant input, as in the previous problem. Alternatively, a simpler design is to use a low-pass filter or a 2-DOF implementation of the PID with the slow zeros in the feedback path. We will only discuss the last two options here.



A simple first order low-pass set-point filter can be designed approximately based on the frequency response of the closed-loop transfer function. We can then iterate on the filter pole to meet the specification:

```
>> p=170; step(c2d(tf([1/500 1],[1/p 1]),.001,'Tustin')*fbk(c2d(P,.001)*dpid,1))
```

The alternative is to move the PID zeros to the feedback path and iterate on the zeros of the cascade part (2-DOF implementation). Unfortunately, while it is straightforward to obtain a non-overshooting design, complex zeros are required to maintain the speed of the response. The figure shows the effect of the cascade filter zeros from 0.8 to 0.95.




```
>> p=-.953,F=tf(conv([1,p],[1,p]),num,.001),F=F/dcgain(F),
>> step(fbk(c2d(P,.001)*dpid*F,inv(F)),'r')
```

Problem 3.

Design a PID controller for the flexible inverted pendulum with transfer function

$$\frac{\{1.478\}}{\{s^2 + \ 0.0635s \ - 19.54\}} + \frac{\{0.000332\ \mathrm{s}^2 + \ 0.3785\ \mathrm{s} \ + \ 177.5\}}{\{\mathrm{s}^2 + \ 15.52\ \mathrm{s} \ + \ 64750\}}$$

For this problem, the PID should be augmented by a low-pass filter to increase roll-off beyond bandwidth and avoid the excessive excitation of the flexible modes. The sampling frequency is 1000Hz and the choice of closed-loop bandwidth is left as a design parameter. Use a 3rd order low-pass filter, with bandwidth roughly at 2x or 3x of the crossover frequency. In your design, include a prefilter to maintain overshoot to step reference changes under 5%. Verify the stability of your controller with simulations.

The filter is needed to attenuate the resonance peak of the flexible mode so that it does not cause the loop magnitude with the PID to exceed unity. At that frequency, the PID will be in its high frequency gain that is expected to be large, since considerable phase lead is required to stabilize the plant. On the other hand, the crossover frequency should be higher than the instability (4.4 rad/s). So we need to determine a sensible filter F to allow us to iterate on crossover/phase margin.

Roughly, the design equation is $\angle P + \angle F + \angle C = -180 + PM$. We expect that the PID zeros do not contribute more than 150 deg and since we are looking at a crossover around 10 rad/s, the ZOH will have a minimal effect. So we can iterate very quickly $\angle P + \angle F \geq -200^o$ $(or\ +160^o)$ and adjust the pole of F so that this inequality holds for some frequencies above 5 rad/s. We arrive at a value of 40 for the filter pole. We then set-up an iteration to compute a reasonable PID tuning:

```
%EEE 481, HW 5, Problem 3

P=tf(1.478,[1  0.0635  -19.54])+tf([0.000332  0.3785  177.5],[1 15.52
64750]);
T=0.001;
wc=[];
while isempty(wc), wc=input('crossover '),end

F=tf(1,[1/40,1]);F=F*F*F;
zoh = T/2*wc*180/pi;

[pid1,cpid]=pidpmtune(wc*1.5,P*F,.001,35+zoh)
dpid=c2d(cpid,.001,'tustin')
Pd=c2d(P*F,.001);

 step(fbk(Pd*dpid,1));pause
 bode(fbk(Pd*dpid,1));pause
 margin(Pd*dpid);pause
```

After comparing the responses of a few controllers with different wc and PM, we select PM = 35 and wc = 7 as the best one. The final controller (C-T, D-T) is:

cpid =

```
  7.572 s^2 + 23.44 s + 18.14
  ---------------------------
       0.001 s^2 + s
```

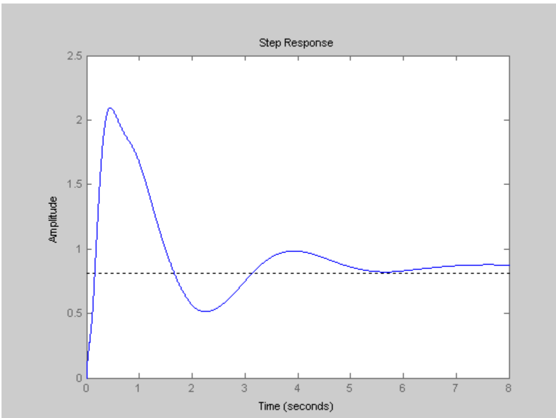
dpid =

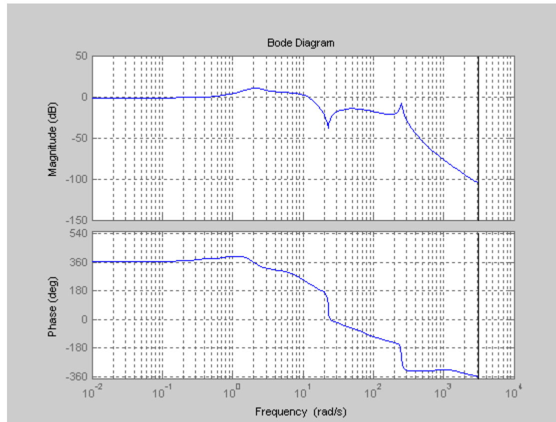
```
  5056 z^2 - 1.01e04 z + 5040
  ---------------------------
```

z^2 - 1.333 z + 0.3333

Sample time: 0.001 seconds

It yields the following step and frequency responses:





The large overshoot is due to the proximity of the RHP pole of the plant to the closed-loop bandwidth. It does necessitate the use of a prefilter.We use a simplified prefilter $\frac{0.05s+1}{as+1}$ and try different values of a to get the overshoot below 5%, arriving at the value $a = 1.7$. The response is not great (significant undershoot) but for the reduced complexity prefilter, it is adequate.

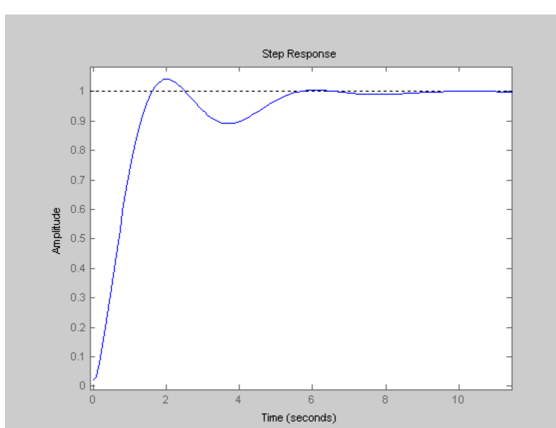
```
>> step((tf([.05 1],[1.7 1]))*fbk(P*cpid,1))
```

The discretized prefilter is

```
 0.0297 z - 0.02911
 ------------------
     z - 0.9994
```

Sample time: 0.001 seconds

(Note: The discrete simulation
>> step(c2d(tf([.05 1],[1.7 1]),.001,'tustin')*fbk(Pd*dpid,1))
diverges due to numerical sensitivity issues. To obtain a correct result P, F, and cpid should all be converted to state-space from the beginning
)



EEE 481 Homework 6

Problem 1.

Consider the pendulum model with input the torque applied at the pivot point and output the angle of the pendulum. (Assume that the pendulum is a rigid rod of length 0.5m, mass 200g evenly distributed, and its rotation around the pivot point is frictionless.)

1. Design a state observer to estimate the angle and angular velocity from noisy angle measurements.
2. Collect 20s of simulation data at 100Hz with random 10Hz excitation around the stable equilibrium such that the amplitude of oscillation does not exceed 6degrees. Implement a 12-bit quantization on the angle measurement for the 360degree range and a 10-bit quantization on the torque for the range [-1, 1]. Formulate the parameter estimation problem and use the batch least-squares algorithm to estimate the parameters of the corresponding transfer function.

Illustrate your findings with a few well-chosen simulations.

EEE 481, Homework 6 SOLUTIONS

Problem 1.
Consider the pendulum model with input the torque applied at the pivot point and output the angle of the pendulum. (Assume that the pendulum is a rigid rod of length 0.5m, mass 200g evenly distributed, and its rotation around the pivot point is frictionless.)

1. Design a state observer to estimate the angle and angular velocity from noisy angle measurements.
2. Collect 20s of simulation data at 100Hz with random 10Hz excitation around the stable equilibrium such that the amplitude of oscillation does not exceed 6degrees. Implement a 12-bit quantization on the angle measurement for the 360degree range and a 10-bit quantization on the torque for the range [-1, 1]. Formulate the parameter estimation problem and use the batch least-squares algorithm to estimate the parameters of the corresponding transfer function.

Illustrate your findings with a few well-chosen simulations.

We start with the pendulum model

$$J\ddot{\theta} = T - \frac{mgL}{2}\sin\theta - \epsilon\dot{\theta}\left|\dot{\theta}\right|$$

Where m is the mass, L is the length, $J = \frac{mL^2}{3}$ is the inertia, and ϵ is the friction coefficient for the pendulum, and $[T, \theta]$ is the I/O pair. The torque T is proportional to the current driving the pendulum motor, but since we have no further data, we will assume a proportionality constant of 1. Linearizing the model around the stable equilibrium $[0, 0]$, we obtain the transfer function

$$P(s) = \frac{60}{s^2 + 29.43}$$

And the state-space realization in terms of angle and angular velocity

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -29.43 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 60 \end{bmatrix} u,$$
$$y = [1 \quad 0]x$$

For the discrete-time model, to be used for state estimation, we find the ZOH equivalent:

$$x_{k+1} = \left(I + 10^{-2}\begin{bmatrix} -0.1471 & 0.9995 \\ -29.42 & -0.1471 \end{bmatrix}\right) x_k + 10^{-2}\begin{bmatrix} 0.2999 \\ 59.97 \end{bmatrix} u_k,$$
$$y_k = [1 \quad 0]x_k$$

For this we define the state observer

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - \hat{y}_k),$$
$$\hat{y}_k = C\hat{x}_k$$

where L is the observer gain which can computed using a variety of approaches. One, particularly attractive method is by using the Kalman Filter equations in their steady-state solution, given by the discrete Riccati equation $L = A\Sigma C^T[C\Sigma C^T + R]^{-1}, \Sigma = A\Sigma A^T + GQG^T - A\Sigma C^T[C\Sigma C^T + R]^{-1}C\Sigma A^T$. While this equation, taken as a recursion, will converge to the steady-state solution, MATLAB also implements efficient numerical methods to solve it:

```
>> L = dlqe(A,G,C,Q,R)
```

Here, G,Q are the input and intensity (covariance) matrices for the state noise and R is the intensity of the output noise. Since we do not have any additional information to model the noise, or optimize specific aspects of the Kalman Filter response, we will simply choose, G = I, Q = BB', and R = a small scalar, to be iterated until a "reasonable" speed of convergence is obtained. For example,

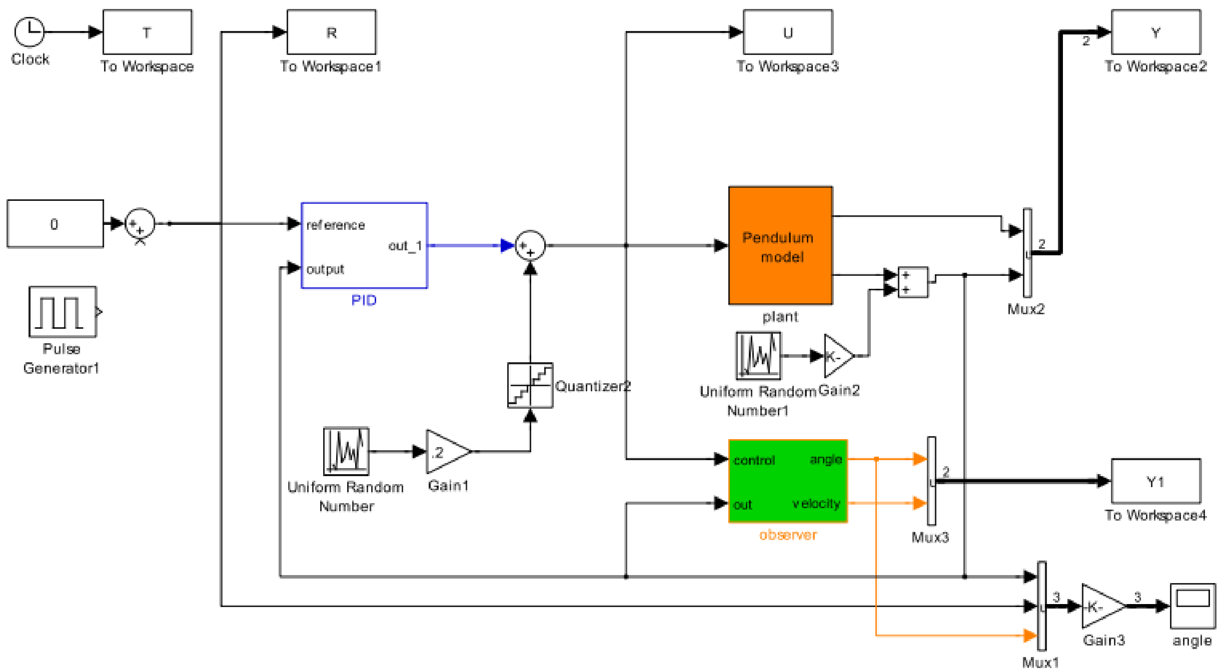
```
>> Hd = c2d(H,0.01)
>> L=dlqe(Hd.a,eye(2,2),Hd.c,Hd.b*Hd.b',0.01),abs(eig(Hd.a-L*Hd.c))
```

This yieds
L = [0.2865, 4.823]

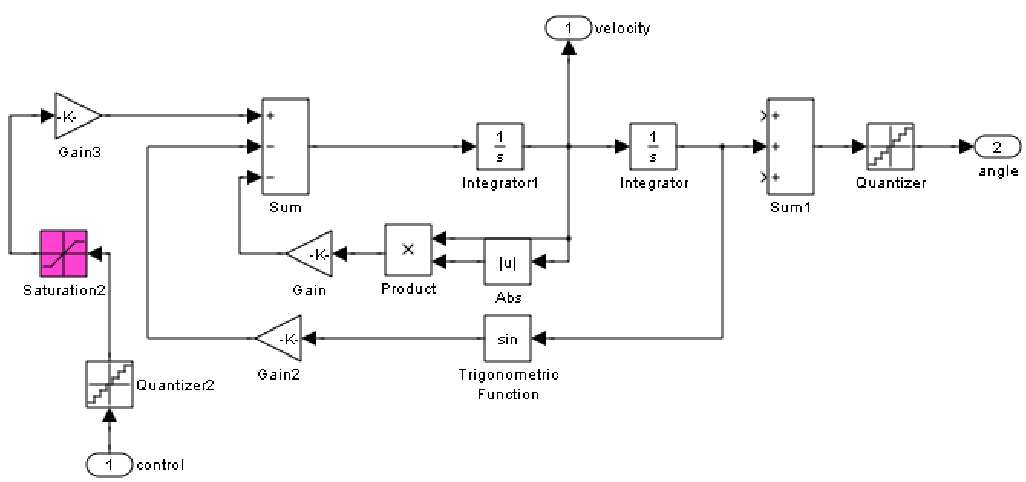
and magnitude of the observer error system eigenvalues 0.873; the latter implies convergence of the error system in 20 samples, or 0.2 sec, which is a reasonable time from a feedback control perspective. (In a quick design, the crossover of the feedback system would be selected around 10-20 rad/s, a factor of 2-4 above the bandwidth of the system poles, both for the stable and the unstable equilibrium case.)

Finally, for implementation purposes, it is often a good idea to use a controller to stabilize the system so that its response stays bounded for any possible test condition. (Especially, for system identification applications.) Omitting the details, here we design a PID to provide 50deg phase margin at 13rad/s:
[Kp, Ki, Kd] = [1.3429e+000 2.6944e+000 1.6398e-001]

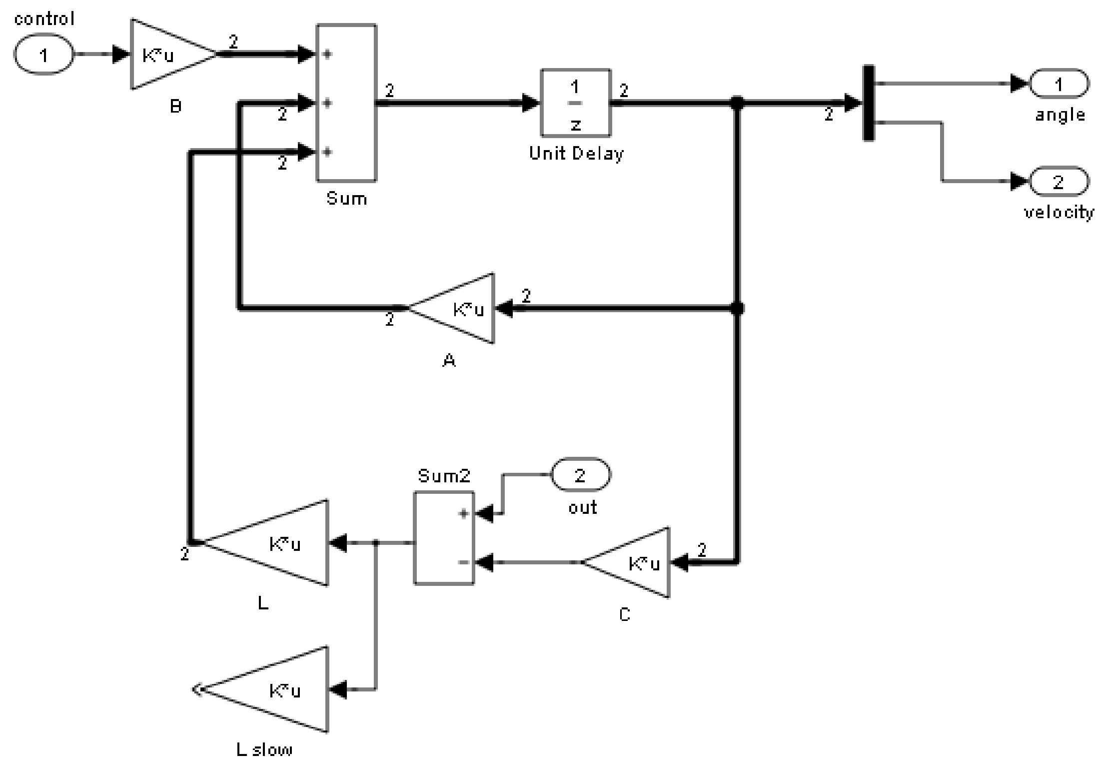
Next, we construct a simulation model to solve the nonlinear pendulum equation, and connect the observer to the system I/O.

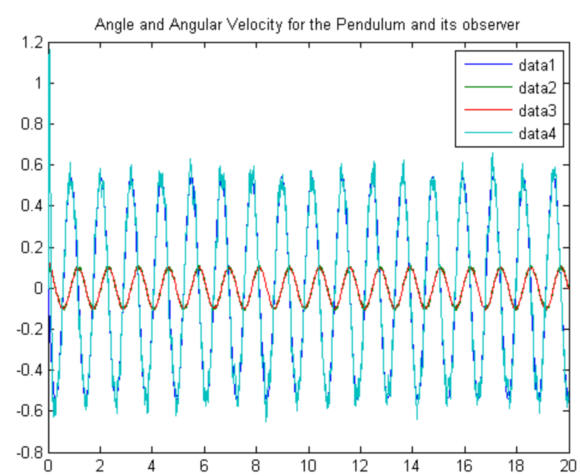


Pendulum Subsystem:



Observer Subsystem:

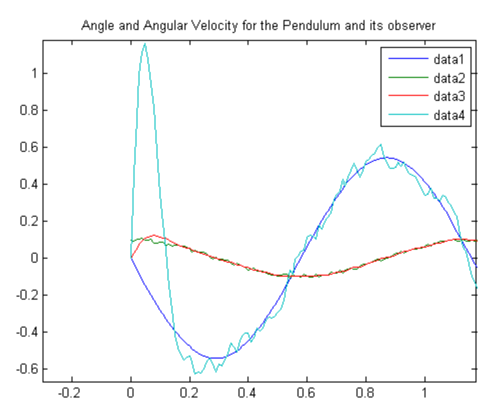




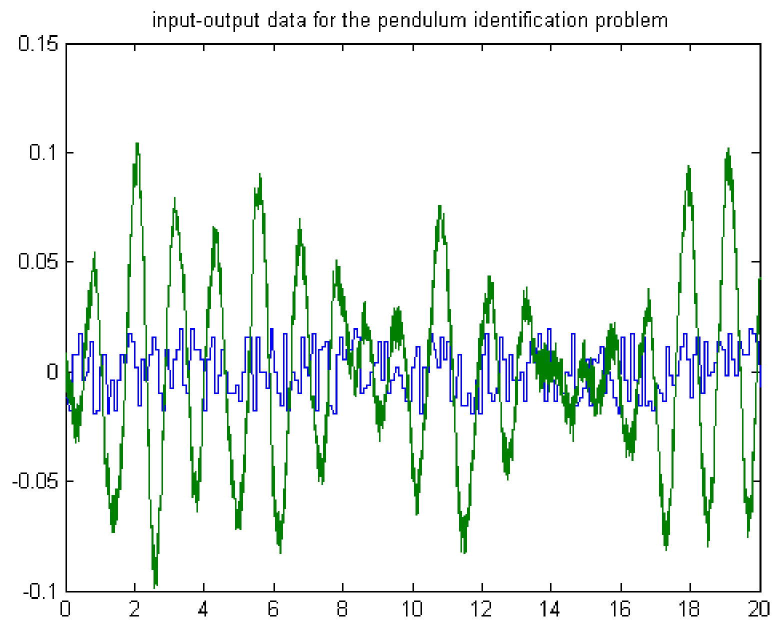
This simulation model allows the study of observer and identification problems under a variety of conditions. We list some below:

- Convergence for different initial conditions (defined in the Pendulum mask)
- Convergence with and without the PID controller, with and without random excitation, with and without output noise
- Use of different observer gains, obtained with different output noise weights (R) in the Riccati equation
- Stable and unstable equilibrium (requires adjustment of the observer model).

Example: Uncontrolled system (for the unstable equilibrium such tests can be performed only for short time intervals), starting with I.C. [0.1, 0]. Here, the angle output is noisy but the velocity is not. Their estimates present a “smoothed” version of the angle, but the velocity estimate is noisy. For a 20s interval, the two traces overlap. With a zoom-in during the initial transient, we can observe the convergence, which takes roughly 0.2s as predicted from the eigenvalues of the observer error subsystem.



For the identification experiment, we connect the excitation at the pendulum input. With zero I.C., and after some trial-and-error we find a gain for the excitation (0.02) which causes the angle deviations to be below 6 degrees. (This is necessary to keep the system near the linearization point where $\sin\theta \simeq \theta$.)



We collect the data (U,Y) and form a regressor for a second order system. For a generalization, we define the filter F (e.g., a delay) and then write the regressor

$$w = [Fy, FFy, Fu, FFu]$$

(for a general case of regressor construction, see a system identification text). Then, the LS approximation problem has a solution

$$q = w\backslash y = (w^T w)^{-1} w^T y$$

From which the identified system can be expressed as

$$H = \frac{q(3)F + q(4)FF}{1 - q(1)F - q(2)FF}$$

The MATLAB implementation of this algorithm is shown below

```
>> F=c2d(tf(1,[.1 1]),.01)
>> w=[lsim(F,Y(:,2)),lsim(F*F,Y(:,2)),lsim(F,U),lsim(F*F,U)];q=w\Y(:,2)
>> Hd=minreal((q(3)*F+q(4)*F*F)/(1-q(1)*F-q(2)*F*F)), H=d2c(Hd)
```

Then

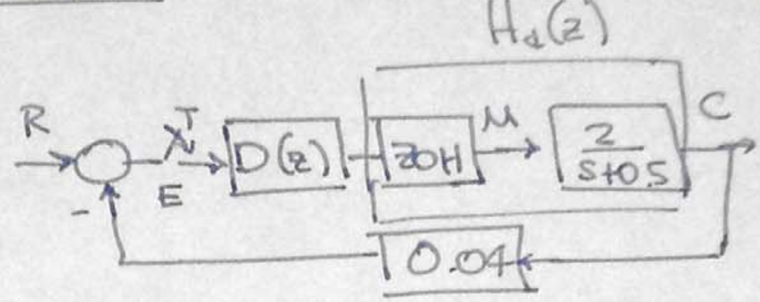
$$H_d(z) = \frac{0.001572\, z \;+\; 0.004358}{z^2 -\; 1.997\, z \;+\; 0.9995}, \qquad H_c(s) = \frac{-0.1394\, s \;+\; 59.33}{s^2 +\; 0.04907\, s \;+\; 29.36}$$

Notice that the model coefficients are fairly "close" to the true linearization (P). However, the identification of the resonance is usually a difficult task and some "smearing" of the peak occurs. A similar result is obtained with the controller in feedback, but now the excitation must be increased by an order of magnitude to achieve the same range of output variation. Otherwise, the output noise causes the signal to noise ratio (SNR) to decrease and the accuracy of the identification deteriorates.

Finally, identification with the pure ARX regressor (delay, F = tf(1,[1 0],.01)) is unsuccessful for this case, because it puts too much emphasis on the high frequencies.

$$H_d(z) = \frac{-0.006659\, z \;+\; 0.02488}{z^2 -\; 0.6264\, z \;-\; 0.3609}$$

Pr. 6.4 System Response:



1. $C(z) = \text{ZOH-Equiv}\left\{\frac{2}{s+0.5}\right\} \times M(z)$

$E(z) = R(z) - 0.04\, C(z)$; note: the sensor is just a gain; no need to compute a separate discretization

$$H_d(z) = \frac{z-1}{z} \mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{\frac{1}{s}\frac{2}{s+0.5}\right\}\Big|_{t=nT} \right\}, \quad T=0.6$$

$$= \text{c2d}(H, 0.6, \text{'zoh'})$$

$$= \frac{1.037}{z-0.7408}$$

Closed loop Response: $C(z) = \dfrac{H_d(z)\,D(z)}{1+H_d(z)\,D(z)\,0.04} R(z)$

With $D(z)=1$, $C(z) = \text{feedback}(Hd, 0.04)$

$$= \frac{1.037}{z-0.6993}$$

Thus, $C(z) = \dfrac{1.037}{z-0.6993}\,\dfrac{z}{z-1}\, r$; r: step amplitude

The problem specifies r: "to command 10° change".

10° change in $C(n)$ will change the sensor reading by 0.4V.

So, ideally, R should change by 0.4.

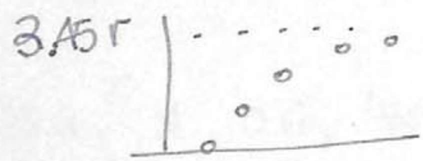
Another interpretation is to find r that will achieve the 10° change in the given system. This would be s.t.

$$10 = \left.\frac{1.037}{z-0.6993}\right|_{z=1} \cdot r \Rightarrow r = 2.9$$

(Either one is acceptable).

The sampled output can be easily computed by

$$\mathcal{Z}^{-1}\left\{\frac{1.037}{z-0.6993}\cdot\frac{z}{z-1}\right\} r = r\left[-2.4\,(0.699)^{n-1} + 3.45\right] u(n-1)$$

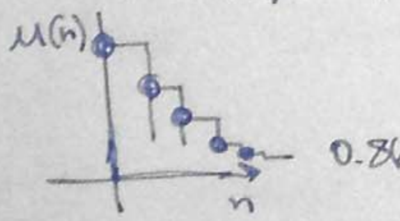


The continuous-time response is more involved and requires the sampled input M(n) which is computed below:

b.) The output of the controller is $\dfrac{D(z)}{1+0.04\,D(z)\,H_d(z)}\,R(z)$

$$= \frac{z-0.741}{z-0.699}\,R(z) = \left(1 - \frac{0.0414}{z-0.699}\right)\frac{z}{z-1}\,r$$

So, $M(n) = r\,u(n) - r\,0.138\,u(n-1) + r\,0.138\,(0.699)^{n-1}u(n-1)$



Now, $M(t)$ is the ZOH output, which is a piecewise constant interpolation of $M(n)$. From that the continuoustime output $C(t)$ can be computed by solving the diff. equ. with initial condition $C(n)$ at $t(n) = nT$ and constant input $M(n)$ in the interval $(nT, nT+T)$. (TEDIOUS!)

c). The steady-state output as $n \to \infty$, $C(n) \to 3.45r$ so for $r = 0.4$, $C(n) = 1.379$. (significant steady-state error, expected since the controller has no integrator).

d). By linearity, the closed loop t.f. is

$$\frac{\dfrac{k\ 1.037/2}{z-0.7408}}{1+\dfrac{k\ 1.037/2\ 0.04}{z-0.7408}} = \frac{0.52\,k}{z-0.74+(0.52K)(0.04)}$$

Its DC gain is $\dfrac{0.52K}{1-0.74+(0.52K)(0.04)}$, approaching $\dfrac{1}{0.04}$ as $k \to \infty$.

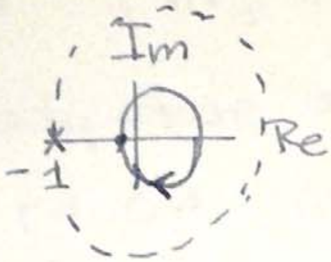
If the system were stable, this would produce 10° output steadystate for an input 0.4 V.

Pr. 7.22 a) For $D(z) = k = 1$, the char. eqn. is

$$1 + D(z) H_d(z) \cdot 0.04 = z - 0.699$$

Its roots are inside the unit circle, hence the system is stable.

b) bode(Hd, nyquist (Hd) produce the derived plots.



0 encirclements

0 open loop poles in RHP

0 closed loop poles in RHP. (stable closed loop)

c) For $k = 1$ the PM is ∞ since the Nyquist plot is completely contained in the unit circle. For GM, we find the Real part of the t.f. when imag(t.f.) = 0. That is -0.0238. So $GM = \frac{-1}{-0.0238} = 42$.

For K to produce $PM = 45°$, we must place the crossover frequency at the point where $\angle H_d = -135°$. Hence, $\omega_{GC} = 3$ rad/s where $|H_d| = -30.4$ dB

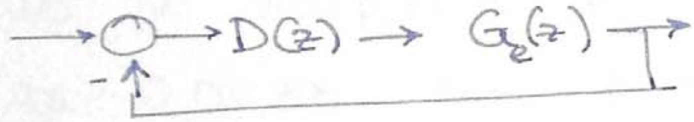
so $K = 33$. (Such a value may or may not exist in general).

d) It is the Gain Margin, $k = 42$

d). The crossover frequency for $H_d = -180$ is 5.24 rad/sec $\left(= \pi/0.6\right)$ since the lowest value of $\angle H_d$ is -180 at π rad/sample or $\pi/0.6$ rad/sec. Therefore, that is the frequency where the closed loop sensitivity approaches infinity $\left(\text{peak} = \frac{1}{\text{distance from } -1}\right)$ and that will be the frequency of oscillation.

From a simulation we verify that the period of oscillation is $\frac{2\pi}{\omega_{oc}} = 1.2$.

<u>PR 8_16</u> With the output defined to be the sensor measurement, we have



$$G_e = \frac{0.0415}{z - 0.741}$$

a) DC gains : forward path $G_e D(1) = \frac{0.0415}{1 - 0.741} = 0.16$

Feedback $\left.\frac{G_E}{1 + G_E}\right|_{z=1} = 0.138.$

b) ss. error $= \frac{1}{1 + k_{DC}} = 0.862 \;(= 1 - 0.138) = 86\%$

c). $D(z) = K$; $G_{CL} = \frac{G_e D}{1 + G_e D} = \frac{0.0415K}{z - 0.741 + 0.0415K}$

For 5% ss. error we want $\frac{1}{1 + k_{DC}} = 0.05 \Rightarrow$

$\Rightarrow K_{DC} = 19 = K \cdot G_e \Big|_{z=1} = K \dfrac{0.0415}{1-0.741}$

$\Rightarrow K = 118.6$

d). The closed loop becomes unstable for $K > 42$ (approx)

$\Rightarrow$ $K = 118.6$ cannot be used to achieve 5% ss error.

e). PI compensator with 45° PM.:

We assume we want to maximize the system Bandwidth ($\approx$ crossover frequency). Since the PI will add phase lag the crossover cannot be higher than $\omega: \angle H_d = -135° = 3$ rad/s. We have freedom to choose the phase of the PI here. We take $\angle PI = -20°$ at crossover to ensure adequate integral action. (we can reevaluate for, say $\angle PI = -10°$ and discuss the differences).

So, $\omega_{GC} : \angle H_d = -115° = 2.12$ rad/s.

$$\angle PI = \angle\left(\frac{\tau s + 1}{s}\right)\Big|_{s=j\omega} = -90 + \tan^{-1}\left(\frac{\tau \omega_{GC}}{1}\right)$$

(Continuous version, use Tustin to discretize later)

$$\Rightarrow \tau = \frac{\tan 70°}{2.12} = 1.3$$

PI : $D(z) = \text{c2d}\left(\dfrac{\tau s + 1}{s}, 0.6, \text{'tustin'}\right) \times K$

where K is the gain required to bring the crossover of $D H_d(z)$ to 2.12.

$\Rightarrow$ PI : $D(z) = \dfrac{29.88z - 18.68}{z-1}$

f,g) There is no ss. error because of the integral part. The simulation shows fast approach to 1 but somewhat slower (5 sec) convergence.

Alternative Computation. (pure D-T).

Let $D(z) = K \dfrac{\tau z - 1}{z-1}$ and, by the same arguments compute τ to yield $-20°$ phase at crossover.

That is $\angle D(z = e^{j\Omega}) = -20$, $\Omega = 2.12 \times 0.6 = 1.272$.

$$\angle D = \tan^{-1} \frac{\tau \sin\Omega}{\tau\cos\Omega - 1} - \tan^{-1} \frac{\sin\Omega}{\cos\Omega - 1} = -20°.$$

$$\Rightarrow \frac{\tau \sin\Omega}{\tau\cos\Omega - 1} = \tan\left[\tan^{-1} \frac{\sin\Omega}{\cos\Omega - 1} - 20°\right] = -3.39$$

$$\Rightarrow \tau = \frac{3.39}{\sin\Omega + 3.39\cos\Omega} = 1.735$$

K for crossover @ 1.272 = 17.5

$$\Rightarrow D(z) = \frac{30.3z - 17.5}{z-1}$$ (very close to the previous sol'n)

Investigate the selection of $\angle PI = -10°$

$\angle H_d(j\omega_{GC}) = -125 \Rightarrow \omega_{GC} = 2.55$

$\angle D(j\omega_{GC}) = -10 \Rightarrow \tan^{-1}\left(\frac{T\omega_{GC}}{1}\right) = -10$

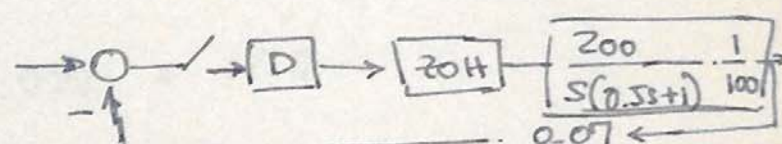
$\Rightarrow T = 2.224$

$\Rightarrow k = 13.1$

$$\Rightarrow D(z) = \frac{33.06z - 25.2}{z-1}$$

(The pure D-T computation yields $34z - 24/z-1$)
This solution has considerable ringing (pole at -0.36) and a slower pole (0.7 vs. 0.56 of the $20°$ PI) even though the crossover is significantly higher (2.55 vs. 2.12). Also, looking at disturbance rejection (at plant input) the $20°$ PI is clearly better at low frequencies.

Pr. 8.19 Robotic arm model



$k = 1, \; T = 0.1$

$$G(z) = \frac{0.0187z + 0.0175}{(z-1)(z-0.819)}$$

By the same reasoning as before, we allocate $20°$ for the PI Lag, yielding a crossover ($\angle G = -115° @$) 0.83 rad/s

Then, $\tau = \dfrac{\tan 70^\circ}{0.83}$

$$D(z) = \text{c2d}\left(\frac{\tau s+1}{s}, 0.1, \text{'tustin'} \right) * K;$$

K for $\omega_{GC} = 0.83$ $\quad \left(|DG(j\omega_{GC})| = 1 \right)$

$$= \frac{0.4287 z - 0.416}{z-1}$$

PD controller : $K(\tau z - 1)$; $DC = K(\tau+1) = 10$

$$\Rightarrow k = \frac{10}{\tau+1}$$

$$|K(\tau z - 1)| \cdot |G(j\omega_c)| = 1$$

$\hookrightarrow z = e^{j\Omega}$; $\Omega = \omega T$

$$\angle \tau z - 1 + \angle G = -135^\circ$$

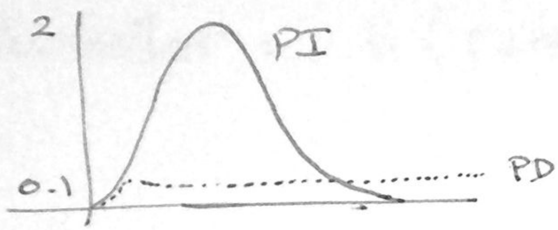
Starting with a guess for wc we can perform an iteration : (NOTE $wc \uparrow \Rightarrow m \uparrow$)

```
wc = 6; [m,p] = bode(G, w), t = -tan((-135-p)*pi/180)
      /(sin(wc*0.1) - tan((135-p)*pi
      /180) * cos(wc*0.1))
d = tf([t, -1], [1], 0.1); K = 1/bode(G*d, w);
d = tf([t, -1], [1], 0.1)*K, m = bode(d, 0).
```

and stop when $m \simeq 10$.

we find $D(z) = 12.98 - 2.93$

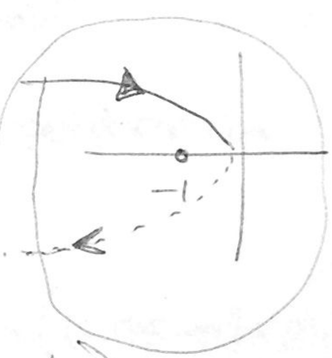
Its step response is much faster than the PI but so is its bandwidth. (1.5 vs 9. rad/s) indicating much higher susceptibility to noise. In terms of input disturbance rejection, it quickly rejects 90% of the disturbance but stays at that steady state (0.1). The PI is much slower, lets the output rise to ~2 but then rejects the disturbance completely



Pr 8.22 $G(z) = \dfrac{0.05(z+1)}{(z-1)^2}$ $T=0.1$

$H = 0.02$ (SENSOR)

a,b) The system cannot be stabilized by a lag compensator because $\angle G$ is always less than -180 and lags will only make the phase more negative. (always +2 encirclements)



c) Applying a similar iteration as before we get that for $\omega = 3.7$ rad/s, $D(z) = 3.142z - 2.151$ with DC gain approx. 1. and $PM = 45°$

d) With this controller $PO \simeq 35\%$, $tr \simeq 0.36$ s.

EEE 481 HW 3

P.1

The read arm on a computer disk drive has transfer function $H(s) = \dfrac{1000}{s^2}$.

1. Design an analog PID controller to achieve a bandwidth of approx. 100Hz with 50deg phase margin.
2. Design a digital PID with a sampling rate 1kHz and simulate the closed loop step response.
3. Keeping the same coefficients of the digital PID, perform a simulation study to determine approximate high/low limits of the sampling rate for which the closed loop is stable.

P.2

Consider the system $x(k+1) = 0.9x(k) + 0.2u(k)$, where the multiplications and the addition are quantized to 0.01. Use simulation to assess the mean, worst-case amplitude, and variance of the error due to quantization (compared to non-quantized operation). Apply various inputs u(k), e.g., random, sinusoid, quantized to 0.01. Compare your results with the theoretical bounds computed from the corresponding transfer functions.

1. mean(x_n) = G(1)mean(n)
2. max|x_n(k)| <= sum|g(k)|max|n(k)|, (g = Z^{-1}{G})
3. var{x_n(k)} <= $|G(e\hat{}j\Omega)|_2^2$var{n(k)}
4. RMS bound: var{x}~RMS^2{x} <= $\max_{\Omega}|G(e\hat{}j\Omega)|^2$ RMS^2{n}
5. Use MATLAB's *"linmod"* command to generate the desired transfer functions directly from Simulink models.

P.3

Ziegler-Nichols Tuning: Apply the two Z-N methods to tune a PID for the plants

$$P_1(s) = \frac{(-0.1s+1)}{s^2+4s+1} \qquad P_2(s) = \frac{(-0.5s+1)}{s^2+0.5s+1}\ .$$

Compare the results with a PID designed for a comparable gain crossover frequency and 50deg. phase margin.

*Hint: Define P as a transfer function object and use step(P) to get an estimate of R,L for the first Z-N tuning. Then iterate k on step(feedback(k*P,1)) until the system is marginally stable (slowly increasing or slowly decreasing response). Then estimate Ku,Pu for the second Z-N tuning. Define the compensators and compare step responses and bode plots for the transfer functions command-to-output and input disturbance-to-output*

EEE 481 Homework 1

Problem 1.

Suppose that we measure a signal 0-5V with a n-bit A/D. What should be the value of n so that

1. The maximum error is less than 1mV?
2. The maximum error is less than 1%?
3. Assuming that the clock used in the A/D conversion is 1MHz, find the maximum conversion time for a successive approximation converter.

For min/max values at the ends of the range, the A/D will have 2^n distinct values dividing the interval. Thus, the resolution is $\frac{5-0}{2^n}$. (If the values are arranged to divide the interval to $2^n - 1$, then the resolution is $\frac{5-0}{2^n-1}$). For a truncating A/D the maximum error is the same, 1 LSB, and for a rounding A/D the maximum error is ½ LSB. Considering the first case and computing the maximum error for different n, we find that we need 13 bits to have error less than 1mV.

The relative error (%) near zero approaches infinity because the quantized conversion is zero but the actual value is positive. This question is not well-posed for the entire interval, but could be more meaningful, if constrained to an interval that does not contain 0.

A successive approximation converter will use roughly 1 clock cycle per bit (since DAC's are much faster than that) and with a bisection algorithm that requires n-steps, so the conversion time is 13us.

Problem 2.

Consider the system $y(k+1) - \frac{1}{8}y(k) = \frac{1}{5}x(k)$, where the multiplications and the addition are quantized to 0.01. Use simulation to assess the mean, worst-case amplitude, and variance of the error due to quantization (compared to non-quantized operation). Apply various inputs x(k), e.g., random, sinusoid, quantized to 0.01. Compare your results with the theoretical bounds computed from the corresponding transfer functions.

1. $\mathrm{mean}(x_n) = G(1)\mathrm{mean}(n)$
2. $\max|x_n(k)| \leq (\sum|g(k)|)\max|n(k)|, (g = Z^{-1}\{G\})$
3. $\mathrm{var}\{x_n(k)\} \leq \left|G(e^{j\Omega})\right|_2^2 \mathrm{var}\{n(k)\}$
4. $\mathrm{var}\{x\} \simeq \mathrm{RMS}^2\{x\} \leq \max^2 |G(e^{j\Omega})|\, \mathrm{RMS}^2\{n\}$

For a round-off quantization, whose mean is 0 LSB, max(|n|) = ½ LSB = 0.005 and var = 1/3(½ LSB)^2 =8.33e-6. RMS(n) = (var{n})$^{1/2}$ = 0.0029.

Realizing the transfer function in terms of delays of the output and input (as shown in the figure below) There are 3 quantization blocks, each one contributing ½ LSB uncorrelated noise to the same summation node. The transfer function from each one is $G(z) = \frac{1}{z-\frac{1}{8}}$, for which, $G(1) = 1.143, \sum|g(k)| = 1.143, \max\left|G\left(e^{j\Omega}\right)\right| = 1.143, \left|\left|G\right|\right|_2 = 1.008$. Evaluating the above estimates (with x_n denoting the error due to quantization)

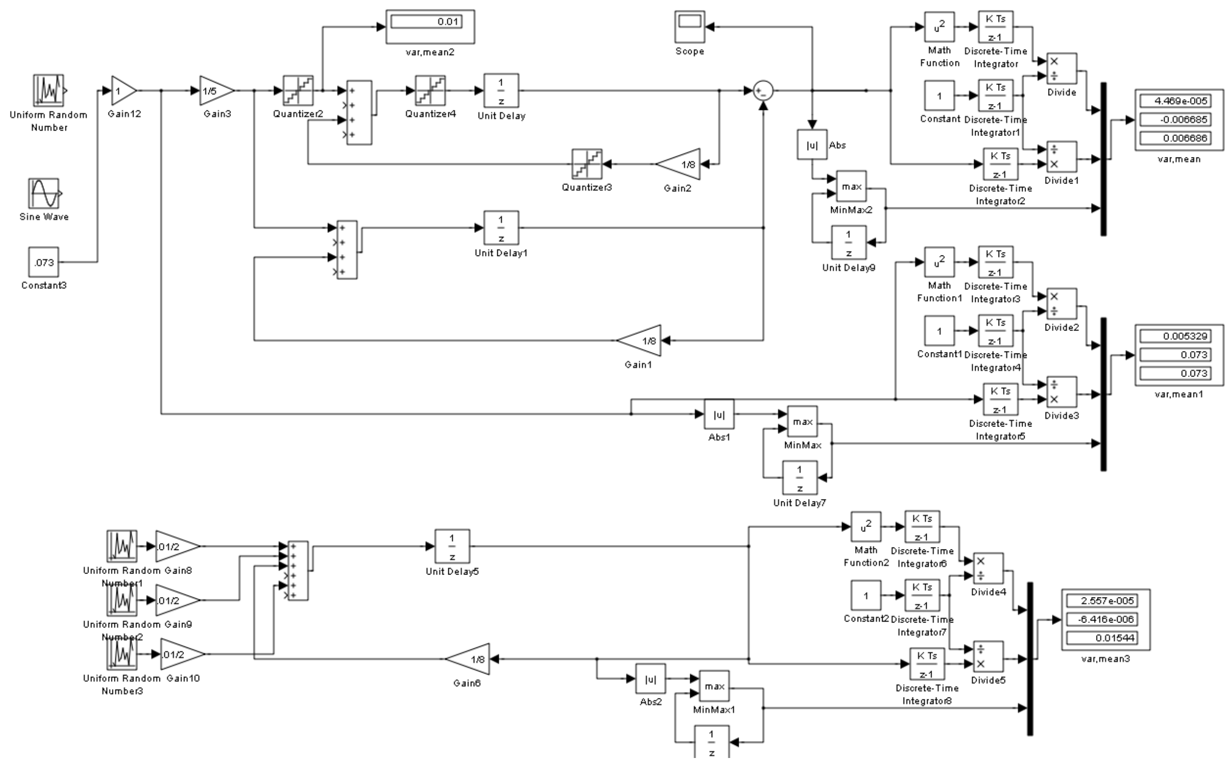
1. $\mathrm{mean}(x_n) = G(1)\mathrm{mean}(n)$ = 1.143 * 3 * 0 = 0 (deterministic noise 1.143*3*0.005 = 0.017)
2. $\max|x_n(k)| \leq (\sum|g(k)|)\max|n(k)|, (g = Z^{-1}\{G\}) = 1.143 * 3 * 0.005 = 0.017$
3. $\mathrm{var}\{x_n(k)\} \leq 3 * \left|G(e^{j\Omega})\right|_2^2 \mathrm{var}\{n(k)\} = 3 * 1.008^2 * 8.33e-6 = 2.54e-5$
4. $\mathrm{var}\{x\} \simeq \mathrm{RMS}^2\{x\} \leq (3 * \max|G(e^{j\Omega})|\, \mathrm{RMS}\{n\})^2 \leq (3 * 1.143 * 0.0029)^2 = 9.9e-5$

Next, we simulate the quantized system, the ideal system, and the system with the noise model of quantizations and tabulate the results as follows:.

	Rand[-1,1]	Const.=0.073	Rand noise model	Theoretical Estimate

Var	1.72e-5	4.47e-5	2.54e-5	2.54e-5 [RMS: 9.9e-5]
Mean	2.27e-5	-0.0067	-1.08e-5	0 [deterministic 0.017]
Max	0.011	0.0067	0.015	0.017 [deterministic 0.017]

Notice that the stochastic variance estimate (using the 2-norm of G) is closer to the observed variance and that the random noise model is fairly representative of the actual errors (for this selection of external inputs). The conservative variance estimate using the RMS deterministic bound (in brackets) is much higher, while the estimate of the maximum amplitude is only conservative by 50%. (This is also because of the specific properties of the system for which sum(|g(k)|) = max|G(e^jw)|.) The deterministic estimates become more accurate for deterministic inputs that expose the worst case. Here a constant 0.073 produced RMS error that was larger than the stochastic estimate. Also note that for the simulation of the random noise model the random number generators must be initialized with different and appropriate seeds so that they produce uncorrelated outputs.



Problem 3.
In a laboratory data acquisition application we would like to use the Diamond MM board to sample several signals at 2kHz and transmit the results to a nearby computer over the RS-232 serial port. How many channels can sample under reasonable assumptions.

The MM has a 12-bit A/D so, without special compression, it will use 2 Bytes per channel. That is, a total of 2*N Bytes per sample time (N = # of channels), or 20*N Bits (assuming one start, one stop, 8-data; other valid protocols are also acceptable). For the transmission to occur under 0.5msec, the rate should be greater than 40N kBaud. Standard rates in that vicinity are 38400, 57600, 115200. So N=1 for 57600 and N=2 for 115200, assuming that the length of transmission is fairly short. The standard has 50 ft for speed 19200 Baud. (The number of channels could go up by 1 if the assumptions are relaxed somewhat, e.g., sue 7 data bits.)

EEE 481 **Homework 2**

Problem 1.

Compute the z-transforms of the following sequences (here u(.) denotes the unit step)

$$u(k+2), \quad \sin\left(\frac{\pi}{10}k+1\right), \qquad \{2-e^{-0.1(k+1)}\}u(k), \qquad 0.9^k u(k-1)$$

$$Z\{u(k+2)\} = z^2 Z\{u(k)\} = \frac{z^3}{(z-1)}$$

For a single-sided transform, (k>=0), $Z\left\{\sin(\frac{\pi}{10}k)\cos 1 + \cos\left(\frac{\pi}{10}k\right)\sin 1\right\} = \frac{z\sin\frac{\pi}{10}\cos 1}{z^2-2z\cos\frac{\pi}{10}+1} + \frac{\left(z^2-z\ \cos\frac{\pi}{10}\right)\sin 1}{z^2-2z\cos\frac{\pi}{10}+1}$.

$$Z\left\{\{2-e^{-0.1(k+1)}\}u(k)\right\} = \frac{2z}{(z-1)} - \frac{ze^{-0.1}}{(z-e^{-0.1})}$$

$$Z\{0.9^k u(k-1)\} = 0.9Z\{0.9^{k-1}u(k-1)\} = 0.9z^{-1}\frac{z}{(z-0.9)} = \frac{0.9}{(z-0.9)}$$

Problem 2.

Solve the difference equation $y(k+2) - \frac{3}{4}y(k+1) + \frac{1}{4}y(k) = x(k) - \frac{1}{5}x(k-1)$ with the initial conditions y(0) = 1, y(-1) = 0 and x(k) = u(k).

One approach is to rewrite the ODE so that the correct initial conditions appear for the shifted outputs. The ODE, shifted by one, now becomes $y(k+1) - \frac{3}{4}y(k) + \frac{1}{4}y(k-1) = x(k-1) - \frac{1}{5}x(k-2)$. Taking transforms and applying the initial condition property, we get

$$zY(z) - zy(0) - \frac{3}{4}Y(z) + \frac{1}{4}z^{-1}Y(z) + \frac{1}{4}y(-1)$$
$$= z^{-1}X(z) + x(-1) - \frac{1}{5}z^{-2}X(z) - \frac{1}{5}z^{-1}x(-1) - \frac{1}{5}x(-2)$$

Substituting the IC and X(z),

$$z^{-1}\left[z^2 - \frac{3}{4}z + \frac{1}{4}\right]Y(z) = z^{-2}\left(z - \frac{1}{5}\right)\frac{z}{z-1}$$

After PFE,

$$Y(z) = \frac{1.6}{z-1} + \frac{-0.8}{z-0.375+0.331j} + \frac{-0.8}{z-0.375-0.331j}$$

Hence,

$$y(k) = 1.6u(k-1) + 2ReZ^{-1}\left\{\frac{-0.8}{z-0.375+0.331j}\right\}$$
$$= 1.6u(k-1) - 1.6Re\{(0.375-0.331j)^{k-1}u(k-1)\}$$
$$y(k) = 1.6u(k-1) - 1.6Re\{(0.375-0.331j)^{k-1}\}u(k-1)$$
$$y(k) = 1.6u(k-1) - 1.6(0.5)^{k-1}Re\ exp\left\{j\ \mathrm{atan}\left(\frac{-0.331}{0.375}\right)(k-1)\ \right\}u(k-1)$$
$$y(k) = 1.6u(k-1) - 1.6(0.5)^{k-1}cos\{0.723(k-1)\}u(k-1)$$

Problem 3.

Consider the system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned} \quad \text{where} \quad A = \begin{bmatrix} -0.2 & 1 \\ 0 & 0.3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C = [0.1 \quad 2]$$

1. Determine whether the system is stable or not
2. Determine whether the system is controllable and/or observable
3. Compute its transfer function
4. Compute the first three samples of its unit-step response.

1. The eigenvalues of A are -0.2 and 0.3, they are inside the unit circle, hence the system is stable.

2. The controllability matrix [B,AB] has rank 2, so the system is completely controllable. The observability matrix [C;CA] has rank 2, so the system is completely observable. In MATLAB, the relevant commands are:
>> G=ss(a,b,c,d,1);
>> Qc=ctrb(G), Qo=obsv(G);

3. The transfer function is $C(zI - A)^{-1}B + D = \frac{2z+0.5}{z^2-0.1z-0.06}$; tf(G)

4. We compute the recursion for the states, starting with x(0)=0 and u(k)=1 for k>=0. Then,
y(0) = 0
y(1) = 2
y(2) = 2.7
y(3) = 2.89
etc. (y = step(G) or y =step(G,0:10))

Problem 4.
Write the differential equation describing the motion of a pendulum with input the torque applied at the pivot point and output the angle of the pendulum. Derive the linearized model around the stable and the unstable equilibria and compute the corresponding transfer functions. Assume that the pendulum is a rigid rod of length 0.5m, with evenly distributed mass 50g, and has a small 100g ball attached to the free end. Its rotation around the pivot point is frictionless.

Newton's law yields, $J\frac{d^2\theta}{dt^2} = -g(m\frac{\mathrm{L}}{2} + \mathrm{ML})\sin\theta + u$, where $J = \frac{1}{3}mL^2 + ML^2$. Hence, substituting the pendulum parameters, $\frac{d^2\theta}{dt^2} = -9.81\frac{15}{7}\sin\theta + \frac{12}{0.35}u = -21.0\sin\theta + 34.3\,u$.

The linearized system around the stable equilibrium has $\sin\theta \simeq \cos 0\,\theta_L, \theta_0 = \theta_0' = u_0 = 0$,

$$\frac{d^2\theta_L}{dt^2} = -21.0\theta_L + 34.3u_L, \qquad \frac{\theta_L(s)}{u_L(s)} = \frac{34.3}{s^2 + 21.0}$$

The linearized system around the unstable equilibrium has $\sin\theta \simeq \cos\pi\,\theta_L, \theta_0 = \pi, \theta_0' = u_0 = 0$,

$$\frac{d^2\theta_L}{dt^2} = 21.0\theta_L + 34.3u_L, \qquad \frac{\theta_L(s)}{u_L(s)} = \frac{34.3}{s^2 + 21.0}$$

EEE 481 Homework 3

Problem 1.

Consider the continuous time system with transfer function $G(s) = \frac{3s}{s^2+9s+4}$.

1. Realize G(s) in state-space and use Forward Euler to compute its discretization, using sampling time T = 0.1. Find the transfer function of the corresponding discrete-time system.
2. Use Forward Euler directly on the transfer function G(s) and compute the corresponding discrete-time transfer function. Realize the discrete-time system in state-space.
3. Compute the first five terms of the discrete-time system impulse response using state-space formulae. Compare with the result of MATLAB's *impulse(.)* function.

1. $\dot{x} = Ax + Bu, y = Cx + Du,\ [A, B, C, D] = \left\{\begin{bmatrix}-9 & -2\\ 2 & 0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, [1.5\ \ 0], [0]\right\}$, is one possible realization. The FE discretization is found by $\dot{x}(T) \simeq \frac{x_{k+1}-x_k}{T} \Rightarrow x_{k+1} = (I + TA)x_k + TBu_k$, $y_k = Cx_k + Du_k$.The transfer function for the discrete time system becomes

$$G_d(z) = C(zI - [I + TA])^{-1}[TB] + D$$

This transfer function can be computed by hand, or by the following MATLAB commands

```
>> G=tf([3],[1,9,4])
>> Gs=ss(G)
>> T=.1;Gd=ss(eye(size(Gs.a))+Gs.a*T,Gs.b*T,Gs.c,Gs.d,T);tf(Gd)

ans =

   0.3z - 0.3
 -----------------
 z^2 - 1.1 z + 0.14

Sample time: 0.1 seconds
```

Notice that the transfer function does not depend on the choice of realization of the continuous transfer function. The above procedure can therefore be used to find the FE discretization of a continuous time system.

2. Performing the substitution $s = \frac{z-1}{T}$, we find exactly the same discrete transfer function as in Part 1. A state space realization is

$x_{k+1} = Ax_k + Bu_k,\ y_k = Cx_k + Du_k,\ [A, B, C, D] = \left\{\begin{bmatrix}0.1 & -0.2\\ 0.2 & 1\end{bmatrix}, \begin{bmatrix}0.2\\0\end{bmatrix}, [1.5\ \ 0], [0]\right\}$,
which does not need to be (and is not) the same as the one in Part 1.

3. We can easily compute the recursion $x_{k+1} = Ax_k + Bu_k,\ y_k = Cx_k + Du_k$, with IC = 0 and $u_k = 1\ for\ k = 0, 0\ oth.$:

```
>> x = [0;0];
>> y=c*x, x=a*x+b;
>> y=c*x, x=a*x;
>> y=c*x, x=a*x;
Etc.
```

We find the values for y: 0, 0.3000, 0.0300, -0.0090, -0.0141, -0.01425
We also find the same values with h = impulse(Gd).

Problem 2.

The first-principles model of a temperature control system is $\dot{Y} = -0.2(Y-273)+Q$, where Y is the Temperature (Kelvin) and Q is the supplied heat (Watts).

1. Use the Forward Euler approximation of derivative $\dot{Y}(t_k) \cong \dfrac{Y(t_{k+1})-Y(t_k)}{T_s}$ to write a corresponding discrete time state-space model for a sampling time of 1sec.
2. What is the discrete-time transfer function of the system?
3. What are the limitations (if any) of this discretization method.

1. $Y_{k+1} - Y_k = -0.2Y_k + 54.6 + Q_k$

2. From Q to Y, the discrete transfer function is $\frac{Y(z)}{Q(z)} = \frac{1}{z-0.8}$. (54.6 can be viewed as an external input, or the output can be interpreted as the incremental output over the equilibrium solution Y = 54.6 for Q = 0.)

3. The stability constraint for the discrete model is $|1-0.2T_s| < 1 \Rightarrow T_s < 10$. Of course, for a sensible approximation, the sampling time should be much less that this bound. E.g., one-half the value will produce a discretized system with pole at the origin, i.e., the entire dynamic response is modeled by a single delay.

Problem 3.

An analog filter with the transfer function $\dfrac{1}{(10s+1)(0.01s+1)}$ is to be replaced by a computer. Determine an appropriate sampling time and the transfer function of the discretized filter. You may use any discretization method you like but you should justify all choices.

A reasonable choice for the sampling time would be related to the system bandwidth (0.0985rad/s ~= 0.1). One may choose different rules of thumb.

- 6 samples/rise time: tr = 2/BW = 20s; T = 20/6 = 3.33s or f = 0.3Hz. (Measuring tr from a step response simulation we find 22.3s which is reasonably close).
- Nyquist = 10 x BW = 1rad/s = 0.16Hz => f = 2*Nyquist = 0.32Hz, T = 3.13s. (This is similar to the above since tr = 2/BW (BW in rad/s) => T = 1/(3BW) = 1/(6piBW) (BW in Hz) => f = 19BW (BW in Hz).)
- ZOH adds 6deg phase lag at BW (a feedback-related spec), wT/2 = 0.1 => T = (0.2/BW)/0.1 = 2s.

Since we are trying to replace an analog filter and have a discretization with similar filtering properties, a Tustin discretization is the more reasonable choice. Thus, for T = 3.33s (not a unique choice), the discretized transfer function is

$$H_d(z) = \frac{0.1419\,\mathrm{z}^2 + 0.2838\,\mathrm{z} + 0.1419}{\mathrm{z}^2 + 0.2735\,\mathrm{z} - 0.7060}$$

One potential drawback of this solution is that it is bi-proper (y_k requires u_k)

However, for a Forward Euler discretization, the sample time is constrained by the fastest sampling constant (0.01). Here, choosing T = 0.01 will transform this term to $0.01\frac{(z-1)}{0.01} + 1 = z$, i.e., it is approximated by a single delay. Since the system response is dominated by the slower mode (10s+1), such an approximation is acceptable (assuming of course that such an oversampling is possible).
For this case,

$$H_d(z) = \frac{0.001}{\mathrm{z}(\mathrm{z}-0.999)}$$

EEE 481 **Homework 4**

Problem 1.

Consider the following system with transfer function $P(s) = \dfrac{-s+3}{(s+0.1)(s+2)}$.

1. Design a PID so that the closed loop crossover is at 0.7rad/s and the phase margin is 50^o.
2. Select a method and the sampling frequency and discretize the PID.

For a discrete design we should first select the sample time so that the ZOH contributes, say, -3deg phase at crossover, i.e., w_cT /2= 0.105/2 or T = 0.15 sec. The phase of P alone at 7 rad/s is -114 deg, so we may be able to use a PI to control it. We define:

$$C(s) = \frac{K(\tau_z s + 1)}{s}$$

To achieve 50 degrees phase margin with the discrete controller, we should compute the PI zero to provide 50+3 deg phase margin. Here, however, the problem asks for 50 degrees PM: $\tan^{-1}\tau_z 0.7 - 90^o - 114 = -130^o \Rightarrow \tan^{-1} 0.7\tau_z = 74$ deg. (PI is really marginal here.) Then, we compute $\tau_z = \frac{3.49}{0.7} = 5.0$. Substituting back to the gain equation $|P(j0.7)C(j0.7)| = 1 \Rightarrow K = 0.093$. Computing the margins for PC we verify the design.

The sampling frequency is now 1/T = 6.7 Hz and the preferred method of discretization of the PID is Tustin, for which we expect a phase margin of 47 deg., since we did not pre-compensate for the ZOH. The controller has the transfer function

$$C_d(z) = \frac{0.475\, z - 0.461}{(z-1)}$$

If we evaluate its margins, it provides a 46.8 degree PM, very close to the expected value. The step and frequency responses are also very close to the continuous time versions.

In Matlab: step(fbk(P*C,1),fbk(c2d(P,.15)*c2d(C,.15,'tustin'),1))

Problem 2.

1. Ziegler-Nichols Tuning: Apply the two Z-N methods from the notes to tune a PID for the plants:

$$P_1(s) = \frac{-0.4s+2}{s^2+4s+3} \quad P_2(s) = \frac{10(-0.4s+2)}{s^2+5s+20}$$

2. Compare the results with a PID designed for a gain crossover frequency of open-loop bandwidth and 45deg. phase margin.

*Hint: Define P as a transfer function object and use step(P) to get an estimate of R,L for the first Z-N tuning. Then iterate k on step(feedback(k*P,1)) until the system is marginally stable (slowly increasing or slowly decreasing response). Then estimate Ku,Pu for the second Z-N tuning. Define the compensators and compare step responses and bode plots for the transfer functions command-to-output and input disturbance-to-output*

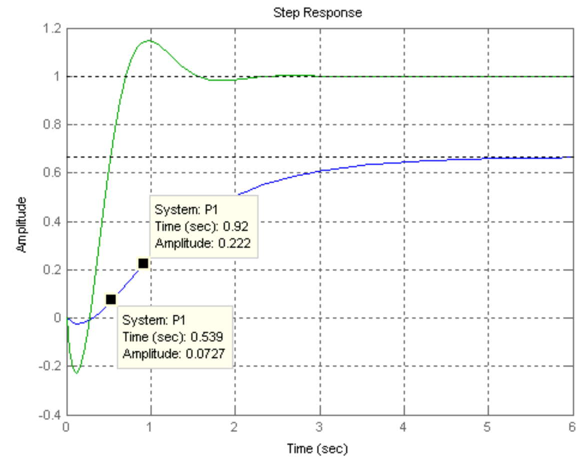
We compute the approximate slopes from the step responses as R1 = 0.39, L1 = 0.36, R2=2.76, L2= 0.28. The corresponding controllers are

$$C_1(s) = \frac{1.3\, s^2 + 8.5\, s + 12.}{s} \quad C_2(s) = \frac{0.18\, s^2 + 1.6\, s + 2.8}{s}$$

For the second method, we try closing the loop with different gains, until oscillatory response is observed. For the first system we find
Ku1 = 10, Pu1 = 1.3 and for the second Ku2 = 1.25, Pu1 = 0.93.
The corresponding controllers are

$$C_1(s) = \frac{0.98\,s^2 + 6\,s + 9.2}{s}$$

$$C_2(s) = \frac{0.087\,s^2 + 0.75\,s + 1.6}{s}$$

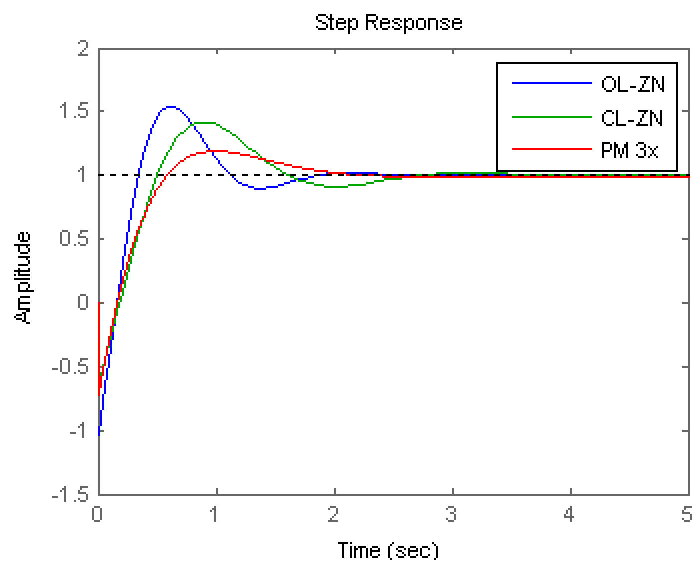


Note that while these gain values happened to produce an exact oscillatory response (due to the round numbers in the system transfer functions), this does not need to be the case in general; for practical applications, sufficient approximation can be obtained by gains that produce decaying oscillations with low damping.

Finally, we design a controller for crossovers at the open-loop bandwidth: The first system has BW = 0.93rad/s and the second has BW = 7.4rad/s. Performing the design, we find

$$C_1(s) = \frac{0.91\,s + 1.8}{s};$$

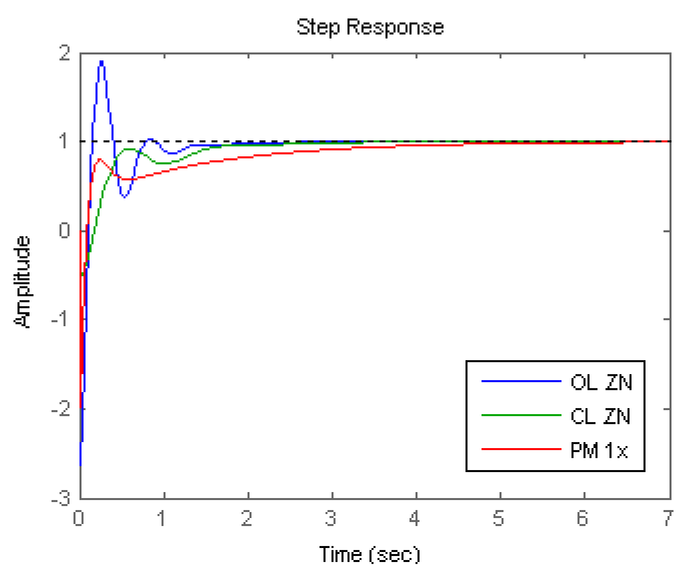
$$C_2(s) = \frac{0.17\,s^2 + 0.82\,s + 0.97}{s}$$



It turns out that the first controller is too slow and not comparable with the Z-N. Instead, we can match the Z-N bandwidth with a crossover at 3x BW:

$$C_1(s) = \frac{1.1\,s^2 + 5.3\,s + 6.4}{s}$$

The step responses with these controllers are shown in the following figures. (Bode plot of loop-tf, step responses. Fbk-ZN: green, OL-ZN: blue, PM-tuned: red). We see that both ZN yield good and similar responses, even though the damping is lower than the 45deg. phase margin controller. They also show more inverse response because their bandwidth is a bit higher.



For the second system the three methods produce different controllers. The PM method is very similar to the CL-ZN if the crossover is taken at 67% of the open-loop BW. But none of the responses is particularly good, showing the difficulty of PID's to manage underdamped systems. The OL-ZN method has a closed loop system whose frequency response stays above 1 without rolling off, so it is a coincidence that the loop is stable. This is an artificial problem since our plant rolls off with only -20dB/dec and we use the ideal improper transfer function for the PID. But even if we add a high frequency roll-off, the difficulty of PID tuning would still remain.

Problem 3.

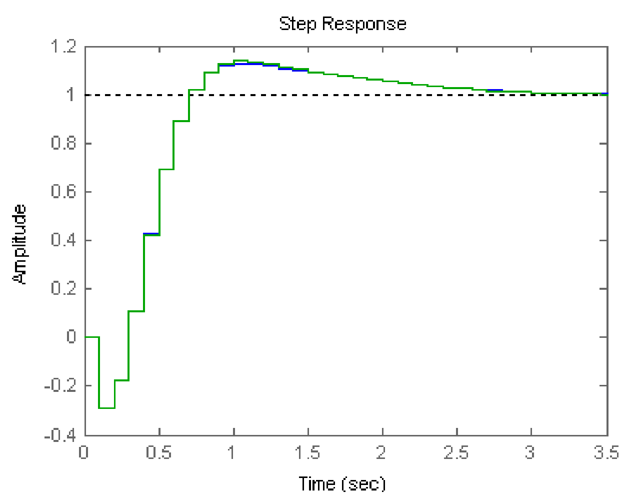
1. Design a PID controller to achieve a bandwidth of 0.5Hz, 50deg phase margin, and to be discretized with a sampling frequency of 10Hz for the system with transfer function
$$P_1(s) = \frac{-0.4s+2}{s^2+4s+3}$$
2. Compare the results with a design in discrete time directly, where the plant is discretized and the parameters of a discrete-time PID are calculated to achieve the same specifications.

We design continuous controller with an additional PM corresponding to the ZOH half-sample delay Tw_c/2. In a first approximation, w_c = BW but a better guess would be w_c = BW/1.5. Selecting the latter, angle(ZOH) = 0.1*(0.5Hz * 6.28/1.5)/2 = 0.1 rad = 6deg. We also select the PID pseudo derivative time constant as 0.05s, anticipating the Tustin transformation, to yield the PID poles at 0 and 1. This is not necessary, but would simplify the derivations of the second part.

The angle required by the two PID zeros is found by
$$\angle P(j\omega_c) + 2\tan^{-1}\tau_z\omega_c - 90 - 6 - \tan^{-1}0.05\times 2.1 = -130$$
$$=> 2\tan^{-1}\tau_z\omega_c = -130+122+90+6+6 = 94 => \tau_z = 0.51$$
The final controller is
$$C(s) = \frac{1.0s^2 + 3.9\,s + 3.9}{0.05\,s^2 + s}$$
Using the Tustin transformation, we obtain the discrete-time controller
$$C_d(z) = \frac{12.1\,z^2 - 19.9\,z + 8.2}{z^2 - z}$$
This controller yields a Phase Margin of 50 deg, at ~2.1 rad/s, as requested.

Next, we consider an entirely discrete time design. We compute the ZOH-equivalent of the plant
$$P_d(z) = \frac{-0.024\,z + 0.0405}{z^2 - 1.65\,z + 0.670}$$
And consider the discrete PID
$$C_d(z) = \frac{K\,(z-a)^2}{z^2 - z}$$
(For consistency, we maintain the same PID poles in the two cases.)
We compute the angle of the plant and the PID poles (dp = tf(1,[1 -1 0],0.1)) at crossover:
>> [m,p]=bode(Pd*dp,2.1), from which p = 124 = -236. (Notice that Matlab requires the continuous frequency as the bode input. The discrete frequency is 2.1*0.1 = 0.21 samples/sec)

For 50 deg phase margin, this requires an angle contribution from each zero of 106/2 = 53 deg.
$$a: \mathrm{atan}\frac{\sin\Omega}{\cos\Omega - a} = 53^o \Rightarrow a = \frac{1.327\times 0.978 - 0.2085}{1.327} = 0.821$$
We then compute the gain K so that the crossover is at 2.1 rad/s, K = 1/ 0.083. The final controller is
$$C_d(z) = \frac{12.1\,z^2 - 19.8\,z + 8.1}{z^2 - z}$$



Obviously, both methods yield very similar controllers and responses. (Any differences are expected to appear much closer to the Nyquist frequency.)

EEE 481 Homework 5

Problem 1.

1. Design a PID controller to achieve a bandwidth of 0.5Hz, 50deg phase margin, and to be discretized with a sampling frequency of 10Hz for the system with transfer function
$$P_1(s) = \frac{-0.1s+1}{s^2+4s+2}$$
2. An additive disturbance enters the plant output with transfer function $P_2(s) = \frac{1}{0.2s+1}$. Design a feedforward component for the PID controller, also discretized at 10Hz, to reduce the effect of the disturbance on the output.

1. We start with a first estimate of the crossover frequency $\omega_c = 0.5 * 6.28/1.5 = 2.1\ rad/s$. At that frequency the plant angle is 242deg = -118deg, the ZOH contribution is $\frac{\omega_c T}{2} = 0.1 * \frac{2.1}{2} = -6$deg. In addition, anticipating a PID controller, the pseudo-differentiator pole contributes $\tan^{-1}\frac{T}{2}\omega_c = -6$deg. Here, we chose the time constant $\tau = T/2$, so that after Tustin discretization the discrete denominator will be $z(z-1)$.

Collecting the angle contributions at crossover, we have
$$2\tan^{-1}\tau_z\omega_c = -180 + 50 + 118 + 6 + 90 + 6 = 90deg$$
The continuous controller (after the gain computation) becomes
$$C(s) = \frac{2.04s^2 + 8.57s + 9.02}{0.05s^2 + s}$$
And its Tustin DT equivalent is
$$C_D(s) = \frac{24.9z^2 \pm 40.3z + 16.3}{z^2 - z}$$
We verify that the last DT controller together with the ZOH equivalent of the plant have PM = 50deg at 2.1rad/s. Its BW is 4.3rad/s which is higher than the initially desired BW (3.14rad/s). If this specification is strict, we could redesign with a crossover frequency 2.1 * 3.14/4.3.

2. We start by recalling the output contributions due to the disturbance

$$y = SP_2[d] - SPH[d]$$
Where H is the feedforward controller and d is the measured disturbance. To design the feedforward controller we can apply a variety of methods.

A. Feedforward at DC: This will only cancel the effect of the disturbance at DC, and by continuity at low frequencies. That will be
$$H = P(0)^{-1}P_2(0)$$

B. H-2 feedforward of plant alone: We consider the minimization
$$\min_H ||P_2 - PH|| = \min_H \left||P_2 - P_i P_o H|\right| = \min_H \left||P_i^{-1}P_2 - P_o H|\right|$$
Where P_i is the inner (stable all-pass) factor of P, which does not alter the norm. P_o is the outer (stable invertible) factor of P. The optimal solution is
$$P_o H = \left(P_i^{-1}P_2\right)_- => H = P_o^{-1}\left(P_i^{-1}P_2\right)_-$$
Where $\left(P_i^{-1}P_2\right)_-$ is the stable projection of $\left(P_i^{-1}P_2\right)$. This solution is valid if H is a proper system, otherwise we need to either multiply with a low-pass filter (an ad-hoc fix) or solve the general problem outlined in the notes that uses a penalty on the control input. For our problem, we can write directly from the transfer functions:

$$P_i = \frac{(-0.1s+1)}{0.1s+1}, \quad P_o = \frac{0.1s+1}{s^2+4s+2}$$

Strictly speaking, this P_o is not outer since it is not biproper. The formal solution would require the approximation of the transfer function by a biproper one, e.g., $P+\epsilon,\ \epsilon \ll 1$, or, $P\left(1+\frac{s}{\epsilon}\right)^N,\ \epsilon \ll 1$, for some N, and then perform the calculations. But here, we can work analytically and wait until the end for such an approximation, if needed.

Now,

$$\left(P_i^{-1}P_2\right) = \frac{0.1s+1}{-0.1s+1}\cdot\frac{1}{0.2s+1} \Rightarrow$$

$$\left(P_i^{-1}P_2\right)_- = \frac{1.667}{s+5} \Rightarrow$$

$$H = \frac{1.667s^2+6.667s+3.333}{0.1s^2+1.5s+5}$$

For which, the Tustin DT equivalent is

$$H_{D1} = \frac{10.71z^2-17.69z+7.156}{z^2-0.993z+0.2}$$

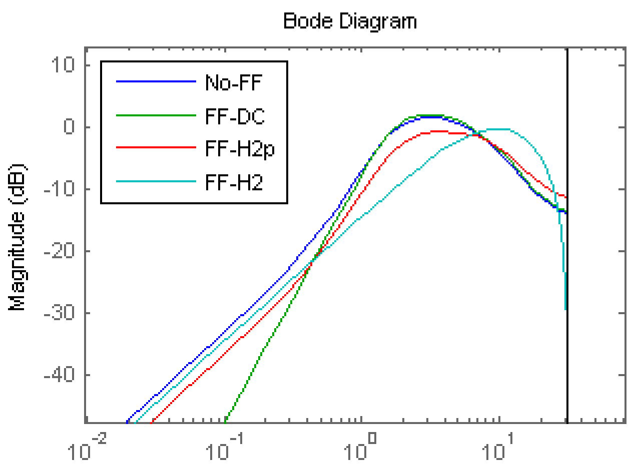
C. The general H-2 FF computation code provided in the past homework solutions performs a similar minimization but accounts for the weight of the sensitivity S.

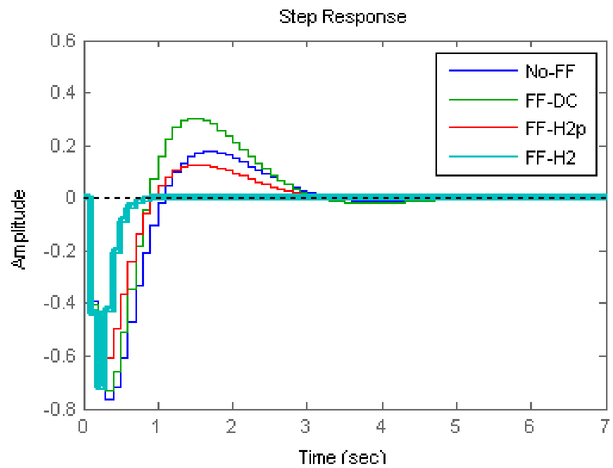
$$\min_H ||SP_2 - SPH||$$

This computation is no longer “easy” like the last one because S contains a differentiator for which an all-pass factor does not exist. A “hack” to get around this problem is to shift the transfer functions so that they contain no zeros on the jw-axis (for CT) or Unit Circle (for DT) but that becomes too complicated to perform without a computational tool.

```
%EEE 481, HW 5, Problem 1
a=-.1
P=tf([a 1],[1 4 2])
Q=tf(1,[.1 1])
[pid,cpid,dpid]=pidpmtune(3.14,P,.05,50+6,2,.1)
Pd=c2d(P,.1)
Qd=c2d(Q,.1)
Sd=feedback(1,Pd*dpid); SPd=feedback(Pd,dpid); SQd=Sd*Qd;
W=1,r=1e-6,W=c2d(tf([.1 1],[1 1e-4]),.1,'tustin');
Gd=[W*SPd;r]; WTd=([W*SQd;0]);
[SPi,SPip,SPo]=iofr(ss(d2c(Gd,'tustin'))); Stil=inv([SPi,SPip]);
R=minreal(Stil*d2c(WTd,'tustin'));
X2=stabproj(R-R.d)+R.d;  H2o=minreal(inv(SPo)*[1 0]*X2);
eig(H2o)
cut=[];
while isempty(cut), cut=input('cut '),end
[H2s,H2f]=slowfast(H2o-H2o.d,cut);H2f=H2f+H2o.d;
HD=c2d(H2f,.1,'Tustin');       % H2 optimal design
```

Evaluating the response of the system with the three different types of Feedforward (DC and H-2plant, H-2), we find that the DC Feedforward does better only in the very low frequencies and then becomes similar (slightly worse) than the No-Feedforward solution. The Feedforward based on the H-2 approximation of the plant (part B) is slightly better in all but the high frequencies (after 10rad/s where the all-pass factor becomes effective). These FF improvements are not very large, because the RHP zero shows up relatively close to the Sensitivity bandwidth where the feedback controller attenuates the disturbance already. On the other hand, the general H-2 optimal solution offers a significant improvement in the mid-range frequencies but it is worse in the high frequencies. In the step responses it shows a much better behavior over all others by correctly accounting for the contribution of all terms in the optimization problem.





Problem 2.

The read arm on a computer disk drive has transfer function

$$H(s) = \frac{1000}{s^2}$$

1. Design an analog PID controller to achieve a bandwidth of approx. 70Hz with 50deg. phase margin.
2. Design a discrete PID for the same bandwidth and phase margin, with a sampling frequency 1kHz and simulate the closed loop step response.
3. What is the maximum bandwidth that can be achieved with a PID having 50deg phase margin and 1kHz sampling?
4. Design a prefilter to achieve overshoot to step reference changes under 5%.

Hint: You need a complete PID for this problem (2-zeros). Use a filter for the pseudo-differentiator with T = 0.001, consistent with the 1ms sampling time.

1. Following our standard design procedure,

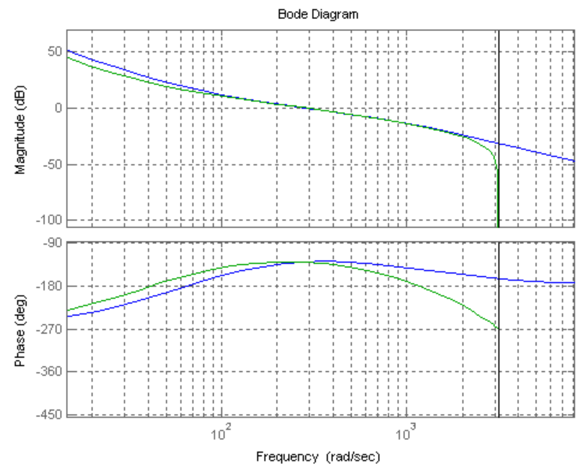
```
P=tf(1000,[1 0 0])
[pid1,cpid1,dpid1]=pidpmtune(70*2*pi,P,.001,50,2)
margin(P*cpid1)
bodemag(feedback(P*cpid1,1))
```

The controller provides the correct phase margin and slightly larger Bandwidth (465 instead of 440rad/s). Its transfer function is

```
Transfer function:
0.2927 s^2 + 35.95 s + 1104
---------------------------
      0.001 s^2 + s
```

2. For the discrete-time design, we adjust the phase margin by the ZOH phase lag $\frac{\omega_c T}{2} = \frac{440}{1.5} \times \frac{0.001}{2}(rad) = 0.147 rad = 8.4^o$

```
[pid2,cpid2,dpid2]=pidpmtune(70*2*pi,P,.001,50+8.4,2,.001)
margin(c2d(P,0.001)*dpid2)
bodemag(feedback(c2d(P,0.001)*dpid2,1))
```



The controller provides the correct phase margin and somewhat larger Bandwidth (553 instead of 440rad/s). Its transfer function is

Transfer function:
208.1 z^2 - 400.1 z + 192.3
----
z^2 - 1.333 z + 0.3333

The system responses are similar but not quite the same. Looking at the Loop transfer function Bode plot reveals that the reason for this, is the somewhat different behavior of the phase around the crossover, even though the margins are identical.

3. The phase margin equation for this system is

$$-180 - 90 - \operatorname{atan}\frac{\omega_c}{1000} - \frac{\omega_c}{2000} \times \frac{180}{\pi} + 2\operatorname{atan}\tau_z\omega_c = -180 + 50$$

Where τ_z is the PID zeros time-constant. At the extreme, the PID zeros are located at the origin, so their time-constant is infinity and this term contributes 2x90=180 deg. Graphical evaluation yields that the maximum possible crossover frequency is 488 rad/s corresponding to BW ~ 732 rad/s or 117 Hz. With a more realistic 160 deg max contribution from the zeros the maximum Bandwidth is limited to only 60Hz.

4. A simple first order low-pass set-point filter can be designed approximately based on the frequency response of the closed-loop transfer function. We can then iterate on the filter pole to meet the specification. After a few iterations, the pole of the filter that meets the specs is found to be -115rad/s.

```
>> p=115; step(c2d(tf([1/500 1],[1/p 1]),.001,'Tustin')*feedback(c2d(P,.001)*dpid2,1))
```

Problem 3.

Design a PID controller for the flexible inverted pendulum with transfer function

$$\frac{\{1.478\}}{\{s^2 + 0.0635s - 19.54\}} + \frac{\{0.000332\, s^2 + 0.3785\, s + 177.5\}}{\{s^2 + 15.52\, s + 64750\}}$$

For this problem, the PID should be augmented by a low-pass filter to increase roll-off beyond bandwidth and avoid the excessive excitation of the flexible modes. The sampling frequency is 1000Hz and the choice of closed-loop bandwidth is left as a design parameter. Use a 3rd order low-pass filter, with bandwidth roughly at 2x or 3x of the crossover frequency. In your design, include a prefilter to maintain overshoot to step reference changes under 5%. Verify the stability of your controller with simulations.

The filter is needed to attenuate the resonance peak of the flexible mode so that it does not cause the loop magnitude with the PID to exceed unity. At that frequency, the PID will be in its high frequency gain that is expected to be large, since considerable phase lead is required to stabilize the plant. On the other hand, the crossover frequency should be higher than the instability (4.4 rad/s). So we need to determine a sensible filter F to allow us to iterate on crossover/phase margin.

Roughly, the design equation is $\angle P + \angle F + \angle C = -180 + PM$. We expect that the PID zeros do not contribute more than 150 deg and since we are looking at a crossover around 10 rad/s, the ZOH will have a minimal effect. So we can iterate very quickly $\angle P + \angle F \geq -200^o$ $(or\ +160^o)$ and adjust the pole of F so that this inequality holds for some frequencies above 5 rad/s. We arrive at a value of 40 for the filter pole. We then set-up an iteration to compute a reasonable PID tuning:

```
%EEE 481, HW 5, Problem 3

P=tf(1.478,[1  0.0635  -19.54])+tf([0.000332  0.3785  177.5],[1 15.52
64750]);
T=0.001;
wc=[];
while isempty(wc), wc=input('crossover '),end

F=tf(1,[1/40,1]);F=F*F*F;
zoh = T/2*wc*180/pi;

[pid1,cpid]=pidpmtune(wc*1.5,P*F,.001,35+zoh)
dpid=c2d(cpid,.001,'tustin')
Pd=c2d(P*F,.001);

 step(fbk(Pd*dpid,1));pause
 bode(fbk(Pd*dpid,1));pause
 margin(Pd*dpid);pause
```

After comparing the responses of a few controllers with different wc and PM, we select PM = 35 and wc = 7 as the best one. The final controller (C-T, D-T) is:

cpid =

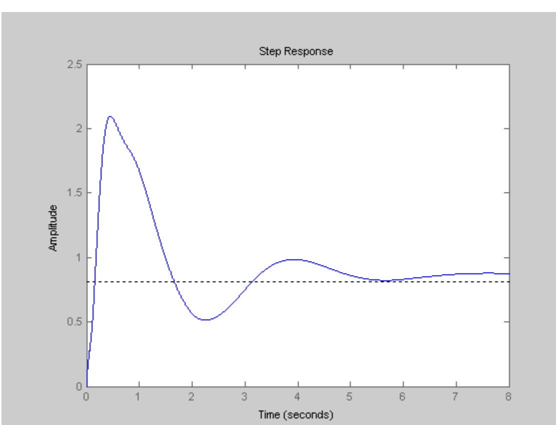
```
 7.572 s^2 + 23.44 s + 18.14
 ---------------------------
       0.001 s^2 + s
```

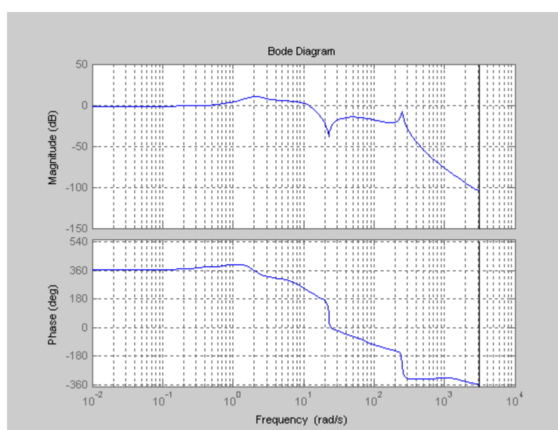
dpid =

```
 5056 z^2 - 1.01e04 z + 5040
 ---------------------------
   z^2 - 1.333 z + 0.3333
```

Sample time: 0.001 seconds

It yields the following step and frequency responses:





The large overshoot is due to the proximity of the RHP pole of the plant to the closed-loop bandwidth. It does necessitate the use of a prefilter.We use a simplified prefilter $\frac{0.05s+1}{as+1}$ and try different values of a to get the overshoot below 5%, arriving at the value $a = 1.7$. The response is not great (significant undershoot) but for the reduced complexity prefilter, it is adequate.

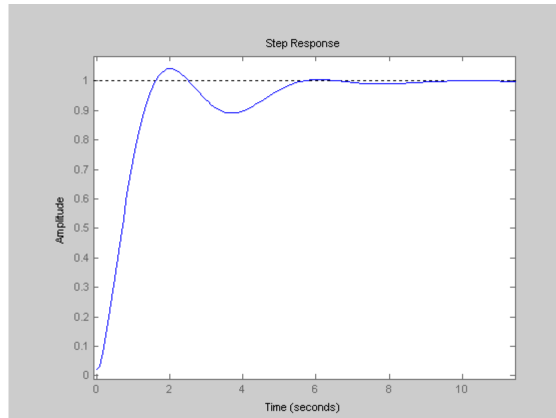
```
>> step((tf([.05 1],[1.7 1]))*fbk(P*cpid,1))
```

The discretized prefilter is

```
 0.0297 z - 0.02911
 ------------------
     z - 0.9994
```

Sample time: 0.001 seconds

(Note: The discrete simulation
>> step(c2d(tf([.05 1],[1.7 1]),.001,'tustin')*fbk(Pd*dpid,1))
diverges due to numerical sensitivity issues. To obtain a correct result P, F, and cpid should all be converted to state-space from the beginning)



EEE 481 Homework 6

Problem 1.
Consider the pendulum model with input the torque applied at the pivot point and output the angle of the pendulum. (Assume that the pendulum is a rigid rod of length 0.5m, mass 275g evenly distributed, and its rotation around the pivot point is frictionless.)

1. Design a state observer to estimate the angle and angular velocity from noisy angle measurements.
2. Collect 20s of simulation data at 100Hz with random 10Hz excitation around the stable equilibrium such that the amplitude of oscillation does not exceed 6degrees. Implement a 12-bit quantization on the angle measurement for the 360degree range and a 10-bit quantization on the torque for the range [-1, 1]. Formulate the parameter estimation problem and use the batch least-squares algorithm to estimate the parameters of the corresponding transfer function.

Illustrate your findings with a few well-chosen simulations.

We start with the pendulum model

$$J\ddot{\theta} = T - \frac{mgL}{2}\sin\theta - \epsilon\dot{\theta}\left|\dot{\theta}\right|$$

Where m is the mass, L is the length, $J = \frac{mL^2}{3}$ is the inertia, and ϵ is the friction coefficient for the pendulum, and $[T, \theta]$ is the I/O pair. The torque T is proportional to the current driving the pendulum motor, but since we have no further data, we will assume a proportionality constant of 1. Linearizing the model around the stable equilibrium $[0, 0]$, we obtain the transfer function

$$P(s) = \frac{43.6}{s^2 + 29.4}$$

And the state-space realization in terms of angle and angular velocity

$$\dot{x} = \begin{bmatrix} 0 & -7.36 \\ 4 & 0 \end{bmatrix} x + \begin{bmatrix} 4 \\ 0 \end{bmatrix} u,$$
$$y = [0 \quad 2.727] x$$

For the discrete-time model, to be used for state estimation, we find the ZOH equivalent:

$$x_{k+1} = \left(\begin{bmatrix} 0.9985 & -0.0735 \\ 0.03998 & 0.9985 \end{bmatrix} \right) x_k + 10^{-2} \begin{bmatrix} 3.998 \\ 0.07998 \end{bmatrix} u_k,$$
$$y_k = [0 \quad 2.727] x_k$$

For this we define the state observer

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - \hat{y}_k),$$
$$\hat{y}_k = C\hat{x}_k$$

where L is the observer gain which can computed using a variety of approaches. One, particularly attractive method is by using the Kalman Filter equations in their steady-state solution, given by the discrete Riccati equation $L = A\Sigma C^T[C\Sigma C^T + R]^{-1}, \Sigma = A\Sigma A^T + GQG^T - A\Sigma C^T[C\Sigma C^T + R]^{-1}C\Sigma A^T$. While this equation, taken as a recursion, will converge to the steady-state solution, MATLAB also implements efficient numerical methods to solve it:

>> L = dlqe(A,G,C,Q,R)

Here, G,Q are the input and intensity (covariance) matrices for the state noise and R is the intensity of the output noise. Since we do not have any additional information to model the noise, or optimize specific aspects of the Kalman Filter response, we will simply choose, G = I, Q = BB', and R = a small scalar, to be iterated until a "reasonable" speed of convergence is obtained. For example,

>> Hd = c2d(H,0.0025)

>> L=dlqe(Hd.a,eye(2,2),Hd.c,Hd.b*Hd.b',0.01),abs(eig(Hd.a-L*Hd.c))

This yieds

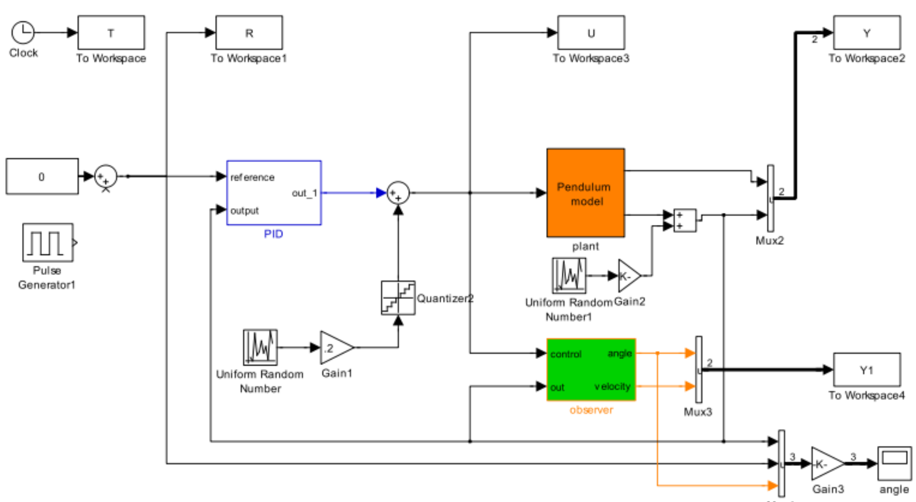
L = [0.4542; 0.1064]

and magnitude of the observer error system eigenvalues 0.872; the latter implies convergence of the error system in 20 samples, or 0.2 sec, which is a reasonable time from a feedback control perspective. (In a quick design, the crossover of the feedback system would be selected around 10-20 rad/s, a factor of 2-4 above the bandwidth of the system poles, both for the stable and the unstable equilibrium case.)

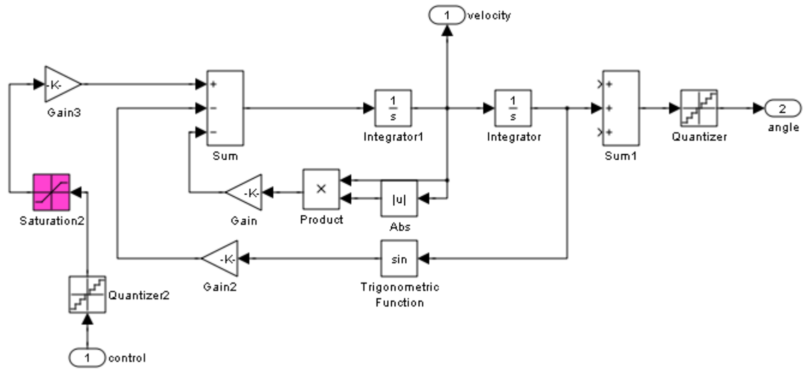
Finally, for implementation purposes, it is often a good idea to use a controller to stabilize the system so that its response stays bounded for any possible test condition. (Especially, for system identification applications.) Omitting the details, here we design a PID to provide 50deg phase margin at 13rad/s:

[Kp, Ki, Kd] = [1.7044e+000 3.3016e+000 2.1153e-001]

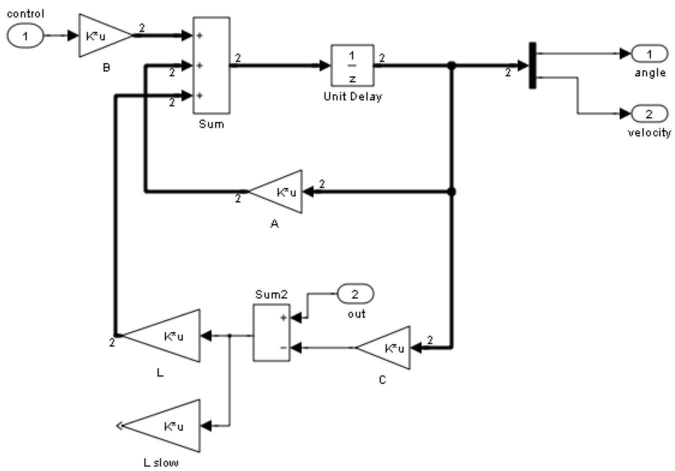
Next, we construct a simulation model to solve the nonlinear pendulum equation, and connect the observer to the system I/O.

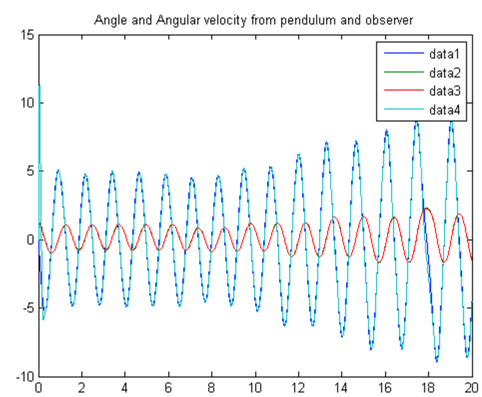


Pendulum Subsystem:



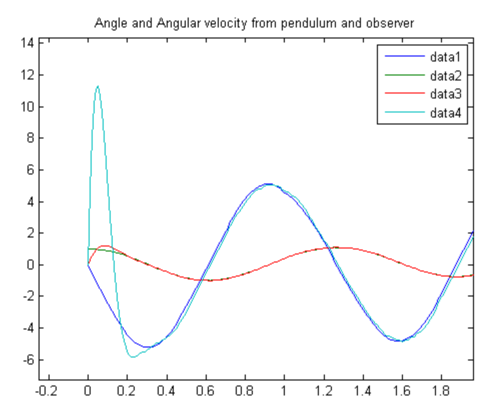
Observer Subsystem:





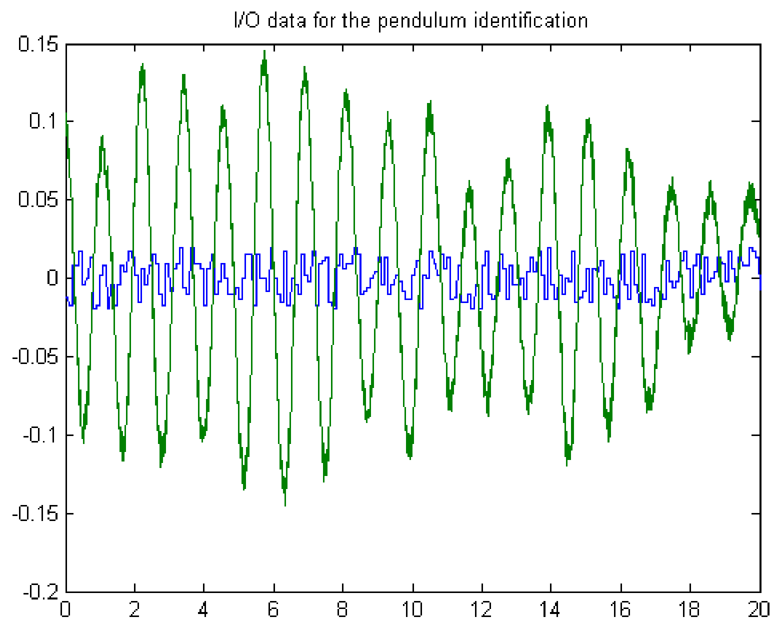
This simulation model allows the study of observer and identification problems under a variety of conditions. We list some below:

- Convergence for different initial conditions (defined in the Pendulum mask)
- Convergence with and without the PID controller, with and without random excitation, with and without output noise
- Use of different observer gains, obtained with different output noise weights (R) in the Riccati equation
- Stable and unstable equilibrium (requires adjustment of the observer model).



Example: Uncontrolled system (for the unstable equilibrium such tests can be performed only for short time intervals), starting with I.C. [1, 0]. Here, the angle output is noisy but the velocity is not. Their estimates present a “smoothed” version of the angle, but the velocity estimate is noisy. For a 20s interval, the two traces overlap. With a zoom-in during the initial transient, we can observe the convergence, which takes roughly 0.3s as predicted from the eigenvalues of the observer error subsystem.

For the identification experiment, we connect the excitation at the pendulum input. With zero I.C., and after some trial-and-error we find a gain for the excitation (0.02) which causes the angle deviations to be below 6 degrees. (This is necessary to keep the system near the linearization point where $\sin\theta \simeq \theta$.)



We collect the data (U,Y) and form a regressor for a second order system. For a generalization, we define the filter F (e.g., a delay) and then write the regressor

$$w = [Fy, FFy, Fu, FFu]$$

(for a general case of regressor construction, see a system identification text). Then, the LS approximation problem has a solution

$$q = w\backslash y = (w^T w)^{-1} w^T y$$

From which the identified system can be expressed as

$$H = \frac{q(3)F + q(4)FF}{1 - q(1)F - q(2)FF}$$

The MATLAB implementation of this algorithm is shown below

```
>> F=c2d(tf(1,[.1 1]),.01)
>> w=[lsim(F,Y(:,2)),lsim(F*F,Y(:,2)),lsim(F,U),lsim(F*F,U)];q=w\Y(:,2)
>> Hd=minreal((q(3)*F+q(4)*F*F)/(1-q(1)*F-q(2)*F*F)), H=d2c(Hd)
```

Then

$$H_d(z) = \frac{-0.00214z \ + \ 0.006692}{z^2 - \ 1.997\, z \ + \ 1}, \qquad H_c(s) = \ \frac{-0.4417\, s \ + \ 45.52}{s^2 + \ -0.03621\, s \ + \ 29.57}$$

Notice that, even though the identified model is unstable, its coefficients are fairly "close" to the true linearization (P), implying that a controller designed for the identified system will also work for the actual system. (The theory behind this statement is "coprime factor perturbations", "gap metric" is discussed in graduate courses.) However, the identification of the resonance is usually a difficult task and some "smearing" of the peak occurs. A similar result is obtained with the controller in feedback, but now the excitation must be increased by an order of magnitude to achieve the same range of output variation. Otherwise, the output noise causes the signal to noise ratio (SNR) to decrease and the accuracy of the identification deteriorates. Finally, identification with the pure ARX regressor (delay, F = tf(1,[1 0],.01)) is unsuccessful for this case, because it puts too much emphasis on the high frequencies.

EEE 481 Test 1 **Name: ___SOLUTIONS_____**

Problem 1:

Find a state-space realization of $\dfrac{0.1}{(z-0.5)(z-0.7)}$

Define $x_k = \begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix}$. Then, using the difference equation $y_{k+2} - 1.2y_{k+1} + 0.35y_k = 0.1u_k$ we write the state update equations $x_{k+1} = \begin{bmatrix} 0 & 1 \\ -0.35 & 1.2 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u_k,\ y_k = [1\ \ 0]x_k + 0u_k$

Problem 2:

What is the steady-state value of the step response of $\dfrac{0.1}{(z-0.5)(z-0.7)}$?

The step response reaches a limit because the transfer function has poles inside the unit circle and the system is stable. From the final value theorem $\lim_{k\to\infty} y_k = \lim_{z\to 1}(z-1)Y(z) = \lim_{z\to 1}(z-1)\frac{0.1}{(z-0.5)(z-0.7)}\frac{z}{z-1} = \frac{0.1}{(0.5)(0.3)} = \frac{1}{1.5} = \frac{2}{3}$.

Problem 2:

Choose the range and number of bits for an A/D converter that will measure a signal with range [-2V, +5V] with 3mV accuracy.

A standard A/D range for this signal is +/- 5V. Then the accuracy of the conversion is $\frac{5V-(-5V)}{2^N} < 0.003V$, where N is the number of bits. Hence, $2^N > 3333.3 => N > \frac{\log(3333.3)}{\log(2)} = 11.7 => N = 12\ bits.$

Problem 3:

An analog filter with the transfer function $\dfrac{1}{(2s+1)(0.01s+1)}$ is to be replaced by a computer. Determine an appropriate sampling time and the transfer function of the discretized filter. You may use any discretization method you like but you should justify all choices.

The filter bandwidth is 0.5rad/s (approx.) so the corresponding Nyquist rate is 1/6.28Hz. A reasonable choice for the sampling time is an order of magnitude faster, that is, 10/6.28Hz, or, T = 0.628s. This sampling time is too large for a simple Forward Euler discretization, because the fast pole at 100 rad/s would result in instability. Instead, we discretize using Backward Euler s = (z-1)/Tz for which the discrete time equivalent is

$$G(z) = \frac{1}{\left(\frac{2(z-1)}{Tz}+1\right)\left(\frac{0.01(z-1)}{Tz}+1\right)} = \frac{T^2z^2}{(2z-2+Tz)(0.01z-0.01+Tz)} = \frac{T^2z^2}{(2.628z-2)(2.638z-0.01)} = \frac{0.235z^2}{(z-0.761)(z-0.0157)}$$

Alternatively, one could choose to use FE for discretization but then the sampling time should be more that an order of magnitude faster, e.g. T = 0.01. This is a poor choice. In this case, $G(z) = \frac{0.005}{(z-0.995)z}$.

Problem 4:

Compute the transfer function of the system with state space representation

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned} \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 \\ -0.5 & 0.2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [0 \quad 0.1]$$

We compute $G(s) = C(zI-A)^{-1}B = [0\ \ 0.1]\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.5 & 0.2 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{0.1z}{z^2-0.2z+0.5}$

EEE 481, Test 1. **Name: __SOLUTIONS__**

75', Closed-book, Closed-Notes. Calculators and One 8 1/5 x 11 sheet (2pages) of notes and formulae allowed.

Problem 1:

Provide brief answers to the following questions:

1. Data from a 0-5V A/D Converter indicate that its resolution (minimum difference between measurements) is approximately 0.3mV. How many bits does this A/D use?
2. How long does it take to transmit 10 integers, 2-bytes each, at 9600 Baud?
3. What is a ZOH equivalent DT system?
4. List three types of quantization errors.

1. 5/2^N (or 2^N-1) should be approximately 0.3e03. Solving, N = 14bits. Notice that if N=15bits, the resolution should have been ~0.15mV.
2. 10x2x(8+2)/9600 = 20.8ms.
3. A discrete time system which produces the same output as the continuous time system at the sample instants, when the input is piecewise constant (switching only at the sample instants). Equivalently, the step response of the ZOH equivalent is the same as the sampled continuous time systems.
4. Type 1: Signal quantization in A/D. Type 2: Addition/multiplication quantization. Type 3: Coefficient quantization in memory.

Problem 2:

An continuous time system is composed of two cascade subsystems, with transfer functions $1/(s+1)$ and $1/(s+2)$. Using a sampling rate of 0.1 sec, determine the transfer functions of:

1. the discrete-time ZOH equivalent system.

2. the discrete-time system obtained by the forward Euler approximation of the derivative ($\frac{dx}{dt}(t_k) \cong \frac{x(t_{k+1}) - x(t_k)}{T}$).

1. First, we compute the step response of the system

$$\frac{1}{(s+1)(s+2)s} = \frac{0.5}{s+2} + \frac{(-1)}{s+1} + \frac{0.5}{s} \Rightarrow y_s(t) = 0.5e^{-2t} - e^{-t} + 0.5 \ (t \geq 0)$$

We sample with T= 0.1

$$y_s(n) = 0.5e^{-2Tn} - e^{-Tn} + 0.5$$

Then take the Z-transform

$$Y_s(z) = \frac{0.5z}{z - e^{-0.2}} - \frac{z}{z - e^{-0.1}} + \frac{0.5z}{z-1}$$

Finally, multiply with (z-1)/z

$$H_D(z) = \frac{0.5(z-1)}{z - e^{-0.2}} - \frac{(z-1)}{z - e^{-0.1}} + 0.5 = \frac{0.004528\,z + 0.004097}{z^2 - 1.724\,z + 0.7408}$$

2.We use the bilinear transformation s = (z-1)/0.1 to find

$$H_D(z) = \frac{0.01}{(z-0.9)(z-0.8)}$$

Problem 3:
Consider the discrete-time system with state space representation

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned} \quad \text{where} \quad A = \begin{bmatrix} 2.1 & -1.08 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$C = [0.5 \quad -0.6]$$

1. Determine its transfer function.
2. Determine whether the system is controllable and observable (hence minimal), and stable.
3. Determine the first 5 samples of its step response, starting with zero initial conditions.

1.

$$H_D(z) = \frac{z - 1.2}{z^2 - 2.1\,z + 1.08}$$

2. $Qc = [B, AB] = \begin{bmatrix} 2 & 4.2 \\ 0 & 2 \end{bmatrix} \quad Qo = [C; CA] = \begin{bmatrix} 0.5 & -0.6 \\ 0.45 & -0.54 \end{bmatrix}$

The controllability matrix is full rank, therefore the system is controllable, the observability matrix has rank 1 so the system is not observable. Hence the system is not minimal. The eigenvalues of the matrix A are 1.2 and 0.9 so the system is not stable.

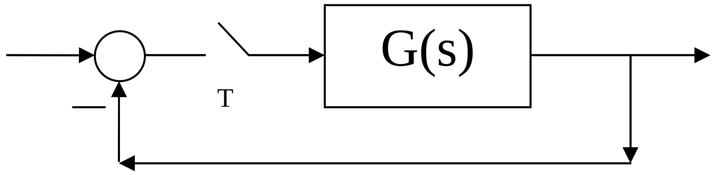
3. From the recursion, y(0) = 0, y(1) = 1, y(2) = 1.9, y(3) = 2.71, y(4) = 3.44, y(5) = 4.10.

EEE 481 **Test 2** **NAME ____SOLUTIONS_____**

Open Book, Closed Notes, 45'

Problem 1

When $G(s) = \frac{2}{(s+1)}$, write the closed loop response for the feedback system, sampled at kT. Find the maximum T for which the system is stable. How is that value related to the Phase Margin of G(s)?



The sampled output response to the reference has the form $Y(z) = \frac{Gzoh}{1+Gzoh}R(z)$, where Gzoh(z) is the ZOH equivalent of G(s):

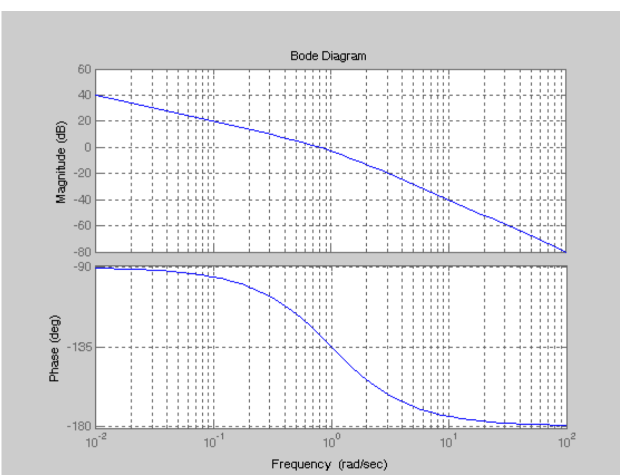
$$Gzoh(z) = \frac{z-1}{z} Z\left\{L^{-1}\left\{\frac{1}{s}\frac{2}{s+1}\right\}\right\} = \frac{z-1}{z} Z\{2[u(kT) - e^{-kT}u(kT)]\} = \frac{z-1}{z} 2\left[\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right] = \frac{2(1-e^{-T})}{z-e^{-T}}$$

The closed loop transfer function is $Gcl(z) = \frac{Gzoh}{1+Gzoh} = \frac{2(1-e^{-T})}{z+2-3e^{-T}}$

The maximum T for stability can be found by setting conditions for the poles of the closed loop transfer function to be inside the unit circle $|2-3e^{-T}| < 1 \Rightarrow e^{-T} > \frac{1}{3} \Rightarrow T < \log\left(\frac{1}{3}\right) = 1.1$

To relate this result to the phase margin of G, we can use the approximation of ZOH as a half-sample delay, contributing $-\omega_{GC}T/2$ phase at crossover. From this, $T < \frac{2PM}{\omega_{GC}}$ is an estimate of the largest sampling time. Here, the crossover frequency is $|G(j\omega_{GC})| = 1 \Rightarrow \omega_{GC} = \sqrt{3} = 1.73$ and the phase margin is $PM = 180 + \arg\{G(j\omega_{GC}) = 180 - \tan^{-1}\sqrt{3} = 2.1(rad)$. So $T < \frac{4.2}{1.7} = 2.5$. The estimate has significant error because the approximation of the ZOH with a delay deteriorates for high frequencies (relative to the delay).

Problem 2



For the system $G(s) = \frac{1}{s(s+1)}$, design a discrete-time PI controller so that the crossover frequency is 0.1 rad/sec and the Phase Margin is 50 deg. Select the discretization time so that the Zero Order Hold contributes 5 deg phase lag at crossover. (For simplicity, you may discretize by any convenient method. Transfer function data may be obtained graphically from the attached Bode plot)

To design the PI, we first compute the phase from the phase margin condition: PM = 180+arg(GC) at crossover. Here, $\arg\big(G(j\omega_{GC})\big) = -90 - \tan^{-1}(0.1)$ and $\arg\big(C(j\omega_{GC})\big) = -90 + \tan^{-1}(\tau\omega_{GC})$ and we need to add -5 deg for the contribution of the ZOH. Hence, $\tau = \frac{\tan(62.1°)}{0.1} = 19$. Then, $K = \frac{1}{|G(j\omega_{GC})|}\frac{\omega_{GC}}{\sqrt{(\tau\omega_{GC})^2+1}} = 4.7e-3$.

The PI controller is now $C(s) = K\frac{\tau s+1}{s}$. To discretize, we select T so that the ZOH angle is -5 deg (approx), i.e., $T = \frac{2}{\omega_{GC}} 5 \frac{\pi}{180} = 1.7$(sec). The discrete controller is $Cd(z) = \frac{0.089z-0.081}{z-1}$ (both Tustin and Euler produce similar coefficients).

EEE 481 Test 2

.1

1. Design and compare a PI and a PID controller to achieve a crossover of 0.02Hz, 50deg phase margin, and discretized with a sampling rate of 0.2Hz for the system with transfer function

$$P_1(s) = \frac{2.2}{(6s+1)(s^2+2.2s+1)}$$

2. Comment on closed-loop bandwidth, disturbance attenuation at 0.002Hz, overshoot and overall set-point tracking capabilities of the two controllers.

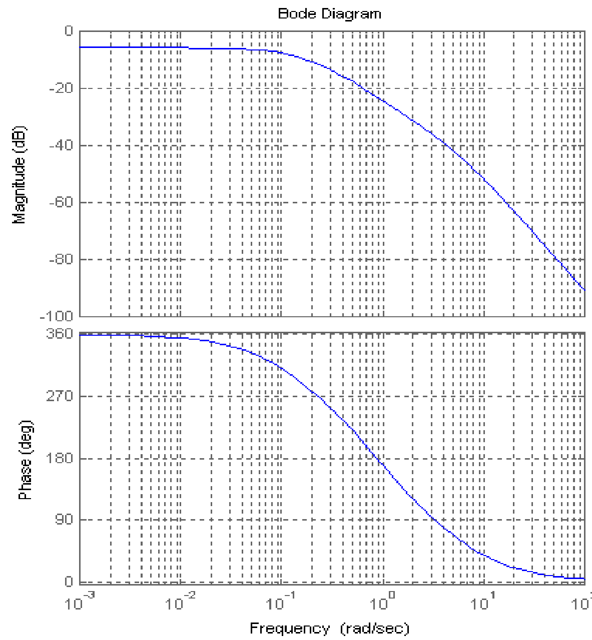
Note: For the PID, use a pseudo-derivative time constant T/2 (half-sample time).

EEE 481 Test 2 NAME: ________________

75’, Closed-book, Closed-Notes, Calculators and One 8 1/5 x 11 sheet (2pages) of notes and formulae allowed.

1. Design a PID controller to achieve a crossover of 0.1 rad/s, 50deg phase margin, for the plant with transfer function (also shown in the adjacent plot)

$$P(s) = \frac{2(-s+1)}{(7s+1)(s^2+6s+4)}$$



For a suitable differentiator time constant, we ask for a phase lag of 0.1 rad at crossover. Hence, $\tau = 1$. The PID is

$$\frac{0.418\,s^2 + 0.636\,s + 0.242}{s^2 + s}$$

2. Select a suitable sample time and discretize the controller. Comment on the phase margin of the discrete time closed loop system.

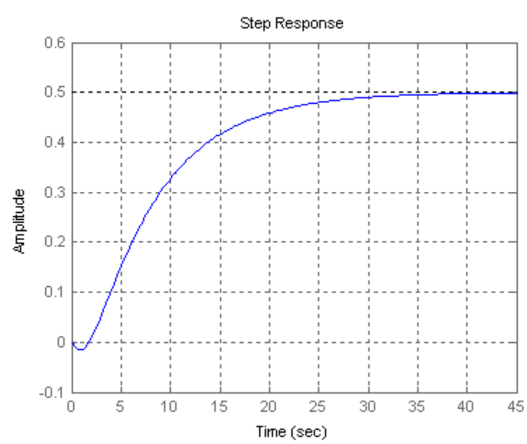
For a suitable sampling time, we ask for a ZOH phase lag of 0.1 rad at crossover. Hence, $T = 2$. Using a Tustin transform, the discretized PID is

$$\frac{0.648\,z^2 - 0.176\,z + 0.012}{z^2 - z}$$

The discretized closed loop will have phase margin approximately 0.1 rad less than the continuous time loop, i.e., ~44deg. (Note: For this problem, a PI controller would be sufficient.)

3. Do you expect that a prefilter will be needed to avoid overshoots in the step response? Briefly describe the design of such a prefilter.

Even though the PID zeros are at 0.76 and are faster than the crossover frequency, the closed-loop will exhibit some overshoot due to the reduced phase margin (less than 60deg). A prefilter can take the form of a low pass filter (or its discrete version), where the zero should be outside the system bandwidth and the pole roughly around the crossover $\frac{1s+1}{10s+1}$. Further refinement can be by trial-and-error. For implementation, and to deal with possible saturating actuator constraints, the filter can be moved inside the loop in a 2-DOF controller configuration.

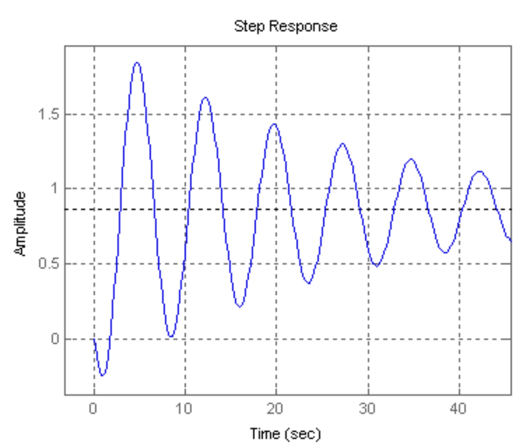


4. Use Ziegler-Nichols rules to design a PID for a system with the step response shown in the adjacent plot.

From the plot, we approximate slope and delay by R=(0.3-0.05)/(8.5-3), L=2, for which the PID is

$$\frac{11\,s^2 + 13.2\,s + 3.3}{s}$$

3. A system is tested in feedback with a proportional controller with gain 1. Use Ziegler-Nichols rules to design a PID controller if the closed-loop step response is as shown in the adjacent plot.



From the plot, we approximate ultimate period by Pu=(12-5), Ku=1 (as given), for which the PID is

$$\frac{0.53\,s^2 + 0.6\,s + 0.17}{s}$$

EEE 481 Test 2 NAME: ________________

75', Closed-book, Closed-Notes, Calculators and One 8 1/5 x 11 sheet (2pages) of notes and formulae allowed.

1. Design a PID controller to achieve a crossover of 2 rad/s, 50deg phase margin, for the plant with transfer function (also shown in the adjacent plot)

$$P(s) = \frac{2(-0.2s+1)}{(7s+1)(s^2+6s+4)}$$

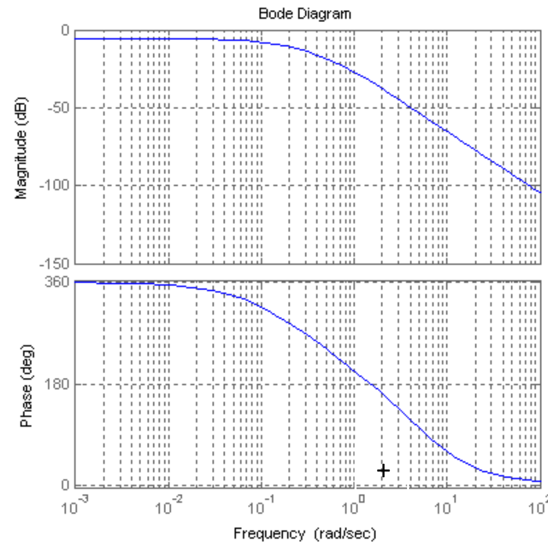
Angle at 2rad/s = 162deg = -198deg =>

$2\angle(\tau_z j\omega + 1) = -180 + 50 + 198 + 90 + 6 = 164 =>$

$\tau_z \omega = 7.1 => \tau_z = 3.56$ (D-filter at $\tau = 1/20$)

Next, we compute the gain of the plant with the controller (so far) at 2rad/s = $\left|\frac{P(j2)(\tau j2+1)^2}{j2(0.05j2+1)}\right| = 0.33$

$$C(s) = \frac{3(3.56s+1)^2}{s(0.05s+1)} = \frac{38.0s^2+21.4s+3}{s(0.05s+1)}$$



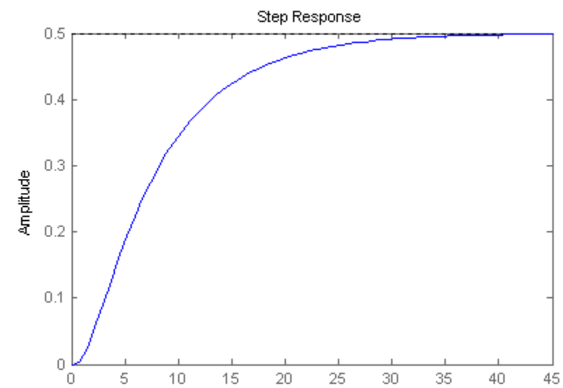
2. Select a suitable sample time and discretize the controller. What will be the phase margin of the discrete time closed loop system?

The Tustin discretization at 1/20s or 20Hz is

$$C_D(z) = \frac{521\, z^2 - 1027\, z + 506.5}{z^2 - 1.333\, z + 0.3333}$$

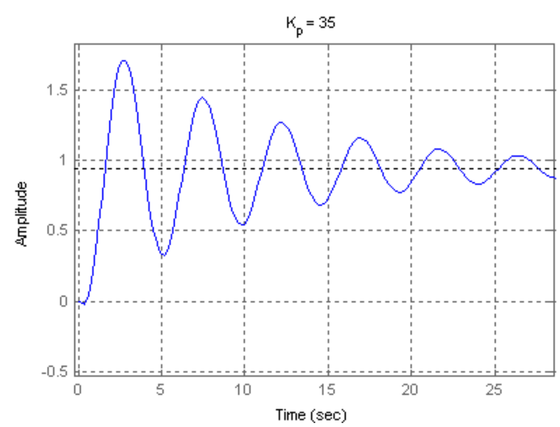
This sampling time is consistent with the D-filter selection, it will add only -3deg lag at crossover but is a little more conservative than the 6-samples-per-rise-time rule (actual BW for this is 5rad/s so this rule would yield T = 1/15sec).

Tustin preserves the phase very well (below 1/3rd of Nyquist) and the ZOH is effectively ½ sample time delay at crossover, implying that the DT PM is 50-3 = 47deg.



3. Do you expect that a prefilter will be needed to avoid overshoots in the step response? Briefly describe the design of such a prefilter.

We expect a significant overshoot because the crossover is well above the open loop BW and a much slower pair of zeros is used in the controller. We should select a lowpass filter to reduce the peaking of the closed loop transfer function. For example, we can start with $F(s) = \frac{(1/2s+1)}{(as+1)}$, where the zero is roughly at closed loop BW and the pole is near the compensator zeros, and iterate until the overshoot is satisfactory.



4. Use Ziegler-Nichols rules to design a PID for a system with the step response shown in the adjacent plot.

We estimate the slope as R = (0.18-0.05)/(5.85-2.15)=0.035 and the delay L = 1.1. Then, the ZN rules yield

$$C(s) = \frac{14\, s^2 + 31\, s + 14}{s}$$

3. A system is tested in feedback with a proportional controller with gain 35 ~~1~~. Use Ziegler-Nichols rules to design a PID controller if the closed-loop step response is as shown in the adjacent plot.

We estimate the Ult. Period as Pu = 7.45-2.75 = 4.7 and the Ku = 35 (from the plot title). Then, the ZN rules yield

$$C(s) = \frac{12\, s^2 + 21\, s + 9}{s}$$

1	P	PI	PID
Kp	1/RL	0.9/RL	1.2/RL
Ki	-	$0.27/RL^2$	$0.6/RL^2$
Kd	-	-	0.5/R

1	P	PI	PID
Kp	0.5Ku	0.45Ku	0.6Ku
Ki	-	0.54Ku/Pu	1.2Ku/Pu
Kd	-	-	0.075KuPu

EEE 481, Test 1. **Name: ______________________**

75', Closed-book, Closed-Notes, Calculators and One 8 1/5 x 11 sheet (2pages) of notes and formulae allowed.

Problem 1. Provide brief answers to the following questions:

1. What is the resolution of a 10bit A/D converter?
2. What is the Baud rate in serial communications?
3. What is a ZOH

1. 10bits means 2^10 levels of conversion, or 1:1024 of the full range.
2. Bits per second.
3. Zero-order Hold, it is a D/A conversion method by which the conversion value is kept constant for one sample period (piece-wise constant, or staircase conversion).

Problem 2:

An analog band-pass filter with the transfer function $\frac{s}{(s+1)(s+100)}$ is to be replaced by a computer. Determine an appropriate sampling time and the transfer function of the discretized filter using any suitable method.

The frequency response should be approximated at least up to 100rad/s to recover the band-pass properties of the filter. We can choose T corresponding to a couple of orders of magnitude above the filter bandwidth (100rad/s), but this choice must also take into account the frequency content of the signals to be filtered. For example, choosing T = 0.001, (6.2kHz) we get, after Euler discretization s = (z-1)/T,

$$H_d(z) = \frac{0.001(z-1)}{z^2 - 1.899z + 0.8991}$$

which has approximately the same frequency response up to 1000rad/s.

Note that, for low sampling rates, the low-pass character of the filter is constrained by the Nyquist frequency; if we choose T=0.01s (620rad/s), and with Euler discretization, the fast pole becomes a delay, i.e., the high-frequency pole is trivialized and the filter becomes just a high pass.

Problem 3:

Consider the discrete-time system with state space representation

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned} \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 \\ -0.5 & 0.1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad C = [-0.1 \quad 2]$$

1. Determine its transfer function.
2. Determine whether the system is controllable, observable, and stable.
3. Determine the first 5 samples of its step response, starting with zero initial conditions.

1. $H(z) = C(zI - A)^{-1}B = \frac{4z - 0.2}{z^2 - 0.1z + 0.5}$

2. Rank([B,AB]) = 2 (det not zero) => the system is completely controllable.

Rank([C;CA]) = 2 (det not zero) => the system is completely observable

Eig(A) = roots (z^2 – 0.1z+0.5) = 0.5 +/- 0.705j which are inside the unit circle (magnitude ~0.71) => the system is stable.

3. $x_0 = 0 \Rightarrow y_0 = 0,\ x_1 = Ax_0 + B.1 = B, y_1 = Cx_1 = CB = 4, x_2 = AB + B, y_2 = Cx_2 = 4.2, y_3 = 2.22, y_4 = 1.922,\ y\_5 = 2.882$

EEE 481 Test 2 NAME: ___________________

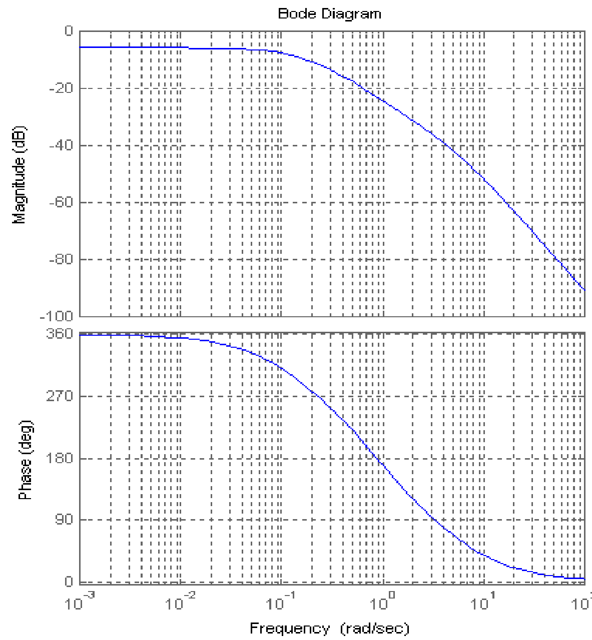
75', Closed-book, Closed-Notes, Calculators and One 8 1/5 x 11 sheet (2pages) of notes and formulae allowed.

1. Design a PID controller to achieve a crossover of 0.1 rad/s, 50deg phase margin, for the plant with transfer function (also shown in the adjacent plot)

$$P(s) = \frac{2(-s+1)}{(7s+1)(s^2+6s+4)}$$

For a suitable differentiator time constant, we ask for a phase lag of 0.1 rad at crossover. Hence, $\tau = 1$. The PID is

$$\frac{0.418\,s^2 + 0.636\,s + 0.242}{s^2 + s}$$



2. Select a suitable sample time and discretize the controller. Comment on the phase margin of the discrete time closed loop system.

For a suitable sampling time, we ask for a ZOH phase lag of 0.1 rad at crossover. Hence, $T = 2$. Using a Tustin transform, the discretized PID is

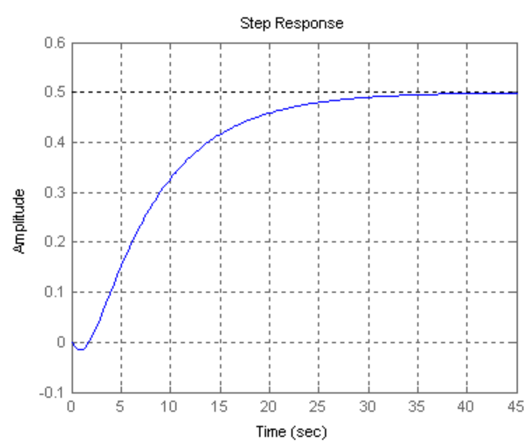
$$\frac{0.648\,z^2 - 0.176\,z + 0.012}{z^2 - z}$$

The discretized closed loop will have phase margin approximately 0.1 rad less than the continuous time loop, i.e., ~44deg. (Note: For this problem, a PI controller would be sufficient.)

3. Do you expect that a prefilter will be needed to avoid overshoots in the step response? Briefly describe the design of such a prefilter.

Even though the PID zeros are at 0.76 and are faster than the crossover frequency, the closed-loop will exhibit some overshoot due to the reduced phase margin (less than 60deg). A prefilter can take the form of a low pass filter (or its discrete version), where the zero should be outside the system bandwidth and the pole roughly around the crossover $\frac{1s+1}{10s+1}$. Further refinement can be by trial-and-error. For implementation, and to deal with possible saturating actuator constraints, the filter can be moved inside the loop in a 2-DOF controller configuration.

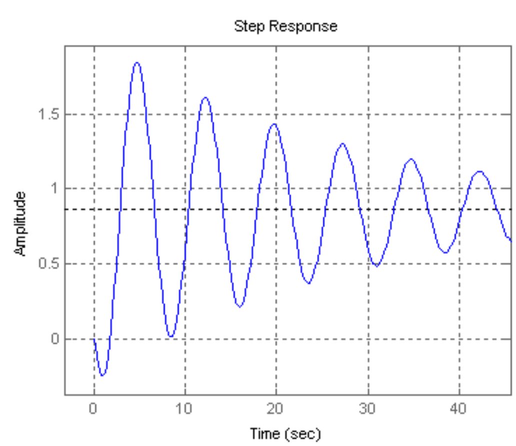
4. Use Ziegler-Nichols rules to design a PID for a system with the step response shown in the adjacent plot.



From the plot, we approximate slope and delay by R=(0.3-0.05)/(8.5-3), L=2, for which the PID is

$$\frac{11\,s^2 + 13.2\,s + 3.3}{s}$$

3. A system is tested in feedback with a proportional controller with gain 1. Use Ziegler-Nichols rules to design a PID controller if the closed-loop step response is as shown in the adjacent plot.



From the plot, we approximate ultimate period by Pu=(12-5), Ku=1 (as given), for which the PID is

$$\frac{0.53\,s^2 + 0.6\,s + 0.17}{s}$$