EEE 582 HW # 1 SOLUTIONS

Problem 1.2

Let λ_i and p_i be the i-th eigenvalue and eigenvector of A, so

$$Ap_i = \lambda_i p_i \tag{1}$$

1.

$$\begin{aligned} A^2 p_i &= AAp_i = A(\lambda_i p_i) = \lambda_i^2 p_i \\ &\vdots \\ A^k p_i &= AA^{k-1} p_i = \lambda_i^k p_i \end{aligned}$$

Hence the eigenvalues of A^k are λ_i^k for $i = 1, ..., n$.

2. Multiply both sides of (1) by A^{-1} (all eigenvalues are different from zero), then

$$A^{-1}Ap_i = A^{-1}\lambda_i p_i \Rightarrow \frac{1}{\lambda_i} p_i = A^{-1} p$$

Hence the eigenvalues of A^{-1} are $\frac{1}{\lambda_i}$ for $i = 1, ..., n$.

3. The eigenvalues of $A^\top$ are given by the roots of

$$\det\left(\lambda I - A^\top\right) = \det\left[(\lambda I - A)^\top\right] = \det\left[(\lambda I - A)\right]^\top$$

and for any square matrix X, $\det(X) = det(X^\top)$, finally

$$\det\left(\lambda I - A^\top\right) = \det(\lambda I - A)$$

Hence the eigenvalues of $A^\top$ are λ_i for $i = 1, ..., n$.

4. Let $A^{\mathrm{H}} = \bar{A}^\top$ (conjugate transpose),

$$\det\left(\lambda I - A^{\mathrm{H}}\right) = \det\left(\lambda I - \bar{A}^\top\right) = \det\left(\lambda I - \bar{A}\right)$$

then

$$\det\left(\lambda I - A^{\mathrm{H}}\right) = \det\left(\lambda I - \bar{A}\right) = \det\left[\overline{\left(\bar{\lambda} I - A\right)}\right] = \overline{\det\left(\bar{\lambda} I - A\right)}$$

Hence the eigenvalues of A^{H} are $\bar{\lambda}_i$ for $i = 1, ..., n$.

5.

$$\alpha A p_i = \alpha (Ap_i) = \alpha \lambda_i p_i$$

Hence the eigenvalues of αA are $\alpha\lambda_i$ for $i = 1, ..., n$.

6. In general the eigenvalues of $A^\top A$ does not relate nicely with the eigenvalues of A. For the special case when $A = A^\top$ (symmetric matrices), the eigenvalues of $A^\top A$ are λ_i^2 for $i = 1, ..., n$.

Problem 1.10

Q symmetric ($Q^\top = Q$), $Q^\top Q = Q^2 \Rightarrow$ eigenvalues of Q^2 are λ_i^2 for $i = 1, ..., n$.

$$\|Q\| = \sqrt{\lambda_{\max}(Q^2)} = \max_i |\lambda_i|$$

From

$$\left|x^\top Q x\right| \le \left\|x^\top Q\right\| \|x\| = \|Qx\| \|x\| \le \|Q\| \|x\| = \max_i |\lambda_i| \, x^\top x$$

hence $\left|x^\top Qx\right| \leq \|Q\|$ for all unit-norm x. Pick x_a as a unit-norm eigenvector of Q corresponding to the eigenvalue that yields $\max_i |\lambda_i|$ (possibly non-unique). Then

$$\left|x_a^\top Q x_a\right| = x_a^\top \left(\max_i |\lambda_i|\right) x_a = \max_i |\lambda_i|$$

thus,

$$\max_{\|x\|=1} \left|x^\top Qx\right| = \|Q\|$$

Problem 1.15

Q symmetric with ϵ_1, ϵ_2 such that

$$0 \leq \epsilon_1 I \leq Q \leq \epsilon_2 I$$

we know

$$0 < \lambda_{\min}(Q)\, x^\top x \leq x^\top Q x \leq \lambda_{\max}()\, x^\top x$$

Pick x as an eigenvalue corresponding to $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$ respectively then

$$\epsilon_1 \leq \lambda_{\min}(Q)\,, \lambda_{\max}(Q) \leq \epsilon_2$$

Therefore

$$\lambda_{\min}\left(Q^{-1}\right) = \frac{1}{\lambda_{\max}(Q)} \geq \frac{1}{\epsilon_2}$$
$$\lambda_{\max}\left(Q^{-1}\right) = \frac{1}{\lambda_{\min}(Q)} \geq \frac{1}{\epsilon_1}$$

For Q^{-1} positive definite

$$0 \leq \frac{1}{\epsilon_2} I \leq Q^{-1} \leq \frac{1}{\epsilon_1} I$$

Problem 1.16

$W(t) - \epsilon I$ is symmetric and positive definite $\forall t$, then for any x

$$x^\top \left(W(t) - \epsilon I\right) x \geq 0 \Rightarrow x^\top W(t) x \geq x^\top \epsilon I x$$

Pick x_t be an eigenvector corresponding to an eigenvalue λ_t or $W(t)$

$$x_t^\top W(t) x_t = \lambda_t x^\top x \geq \epsilon x^\top x$$

That is $\lambda_t \geq \epsilon$. This holds for any eigenvalue of $W(t)$ and every t. Since the determinant is the product of the eigenvalues then

$$\det\left(W(t)\right) \geq \epsilon^n > 0$$

Problem 1.17

Since $A(t)$ is continuously differentiable and invertible for each t, we can write $A^{-1}(t)A(t) = I$, then taking the derivative with respect to time on both sides of the equation

$$\begin{aligned} \frac{d}{dt}\left(A^{-1}(t)A(t)\right) &= \frac{d}{dt} I \\ \dot{A}^{-1}(t)A(t) + A^{-1}(t)\dot{(t)} &= 0 \\ \dot{A}^{-1}(t) &= -A^{-1}(t)\dot{A}(t)A^{-1}(t) \end{aligned}$$

EEE 582 HW # 2 SOLUTIONS

Problem 2.1

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_0(t)y(t) = b_0(t)u(t) + b_1(t)u^{(1)}(t)$$

Let $x_n(t) = y^{(n-1)}(t) - b_1(t)u(t)$ and $x_1(t) = y(t)$ then

$$\begin{aligned}
\dot{x}_1(t) &= \dot{y}(t) = x_2(t) \\
\dot{x}_2(t) &= \ddot{y}(t) = x_3(t) \\
&\vdots \\
\dot{x}_{n-2}(t) &= x_{n-1}(t) \\
\dot{x}_{n-1}(t) &= x_n + b_1(t)u(t) \\
\dot{x}_n(t) &= y^{(n)}(t) - b_1^{(1)}u(t) - b_1(t)u^{(1)}(t) \\
&= -\sum_{i=0}^{n-1} a_i(t)y^{(i)}(t) + b_0(t)u(t) - b_1^{(1)}(t)u(t) \\
&= -\sum_{i=0}^{n-2} a_i(t)x_i(t) + a_{n-1}(t)\left(x_n(t) + b_1(t)u(t)\right) + \left(b_0(t) - b_1^{(1)}(t)\right)u(t)
\end{aligned}$$

then we can write

$$\begin{aligned}
A(t) &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0(t) & -a_1(t) & \cdots & -a_{n-1}(t) \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_1(t) \\ b_0(t) - b_1^{(1)}(t) + a_{n-1}(t)b_1(t) \end{bmatrix} \\
C(t) &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \quad D(t) = 0
\end{aligned}$$

Problem 2.2

$$y^{(n)}(t) + a_{n-1}t^{-1}y^{(n-1)}(t) + a_{n-2}t^{-2}y^{(n-2)}(t) + \cdots + a_1 t^{-n+1}y^{(1)}(t) + a_0 t^{-n}y(t) = 0$$

Let $x_1(t) = t^{n-1}y(t)$, then

$$\dot{x}_1(t) = (1-n)t^{-1}x_1(t) + t^{-1}x_2(t)$$

with $x_2(t) = t^{-n+2}y^{(1)}(t)$, so

$$\dot{x}_3(t) = (2-n)t^{-1}x_2(t) + t^{-1}x_3(t)$$

with $x_3(t) = t^{-n+3}y^{(2)}(t)$, ...

$$\dot{x}_{n-1}(t) = -t^{-1}x_{n-1}(t) + x_n(t)$$

with $x_n(t) = y^{(n)}(t)$, finally

$$\dot{x}_n(t) = -a_{n-1}t^{-1}x_n(t) - a_{n-1}t^{-1}x_{n-1} - \cdots - a_1 t^{-1}x_2(t) - a_0 t^{-1}x_1(t)$$

and we can write $\dot{x}(t) = t^{-1}Ax(t)$, with

$$A = \begin{bmatrix} 1-n & 1 & \cdots & 0 & \\ 0 & 2-n & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

Problem 2.3

$$\ddot{y}(t) + \frac{4}{3}y^3(t) = -\frac{1}{3}u(t)$$

with initial conditions $y(0) = 0$, $\dot{y}(0) = 1$ and $u(t) = \tilde{u}(t) = \sin(3t)$. We can write $\sin(3t) = 3\sin(t) - 4\sin^3(t)$, then the differential equation is

$$\ddot{y}(t) + \frac{4}{3}y^3(t) = \frac{4}{3}\sin^3(t) - \sin(t)$$

Propose as a solution $y(t) = A\sin(t)$, substitute in the differential equation

$$-A\sin(t) + \frac{4}{3}A^3\sin^3(t) = \frac{4}{3}\sin^3(t) - \sin(t) \quad \Rightarrow \quad A = 1$$

and it also satisfies the initial conditons

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad \sin(0) = 0 \\ \dot{y}(0) = 1 \quad &\Rightarrow \quad \cos(0) = 1 \end{aligned}$$

so $y(t) = \sin(t)$ is a solution to

$$\ddot{y}(t) + \frac{4}{3}y^3(t) = -\frac{1}{3}\tilde{u}(t)$$

Let $x_1 = y(t)$, $x_2 = \dot{y}(t)$,

$$\dot{x} = \begin{bmatrix} x_2(t) \\ -\frac{4}{3}x_1^3(t) - \frac{1}{3}u(t) \end{bmatrix} = f(x,u)$$

The linearization around the nominal solution

$$\begin{aligned} A(t) &= \left.\frac{\partial f(x,u)}{\partial x}\right|_{x^*,u^*} = \left.\begin{bmatrix} 0 & 1 \\ -4x_1^2 & 0 \end{bmatrix}\right|_{\substack{x_1=\sin(t) \\ u^*=\sin(3t)}} = \begin{bmatrix} 0 & 1 \\ -4\sin^2(t) & 0 \end{bmatrix} \\ B &= \left.\frac{\partial f(x,u)}{\partial u}\right|_{x^*,u^*} = \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix} \end{aligned}$$

Finally

$$\begin{aligned} \dot{x}_\delta(t) &= A(t)x_\delta(t) + Bu_\delta(t) \\ y_\delta(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_\delta(t) \end{aligned}$$

with $x_\delta(t) = x(t) - \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$, $x_\delta(0) = x(0) - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $u_\delta(t) = u(t) - \sin(3t)$ and $y_\delta(t) = y(t) - sin(t)$

Problem 2.8

Identity dc-gain means that for a given $\tilde{u}$, $\exists\tilde{x}$, such that $A\tilde{x} + B\tilde{u} = 0$, $C\tilde{x} = \tilde{u}$, this implies that the matrix $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ is invertible.

1. If $K \in \mathbb{R}^{m\times n}$ is such that $(A + BK)$ is invertible, then $C(A + BK)^{-1}B$ is invertible.

   Since $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ is invertible , for any K, $\begin{bmatrix} A+BK & B \\ C & 0 \end{bmatrix}$ is invertible, this from

   $$\begin{bmatrix} A+BK & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ K & I \end{bmatrix}$$

   Then

   $$\begin{bmatrix} A+BK & B \\ C & 0 \end{bmatrix}\begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

so

$$\begin{aligned}
(A+BK)R_1 + BR_3 &= I \\
(A+BK)R_2 + BR_4 &= 0 \quad \Rightarrow \quad R_2 = -(A+BK)^{-1}BR_4 \\
CR_1 &= 0 \\
CR_2 &= I \quad \Rightarrow \quad -C(A+BK)^{-1}BR_4 = I
\end{aligned}$$

hence $C(A+BK)^{-1}B$ is invertible.

2. We need to show that there exits N such that

$$\begin{aligned}
0 &= (A+BK)\tilde{x} + BN\tilde{u} \\
\tilde{u} &= C\tilde{x}
\end{aligned}$$

The first equation gives $\tilde{x} = -(A+BK)^{-1}BN\tilde{u}$. Thus we need to choose N such that $-C(A+BK)^{-1}BN = \tilde{u}$. From part 1., we take $N = \left[-C(A+BK)^{-1}B\right]^{-1}$.

Problem 2.10

For $u(t) = \tilde{u}$, $\tilde{x}$ is a constant nominal if and only if $0 = (A + D\tilde{u})\tilde{x} + b\tilde{u}$. This holds if and only if $b \in \text{Im}[A + D\tilde{u}]$, that is, if and only if $\text{rank}(A + D\tilde{u}) = \text{rank}\left[\begin{array}{cc} A + D\tilde{u} & b \end{array}\right]$

If $A + D\tilde{u}$ is invertible, then

$$\tilde{x} = -(A + D\tilde{u})^{-1}b\tilde{u}$$

If A is invertible, then by continuity of the determinant $\det(A + B\tilde{u}) \neq 0$ for all $\tilde{u}$ such that $|\tilde{u}|$ is sufficiently small, equation () defines a corresponding constant nominal. The linerized state equation is

$$\begin{aligned}
\dot{x}_\delta(t) &= (A + D\tilde{u})x_\delta(t) + [b - D(A + D\tilde{u})^{-1}b\tilde{u}]u_\delta(t) \\
y_\delta(t) &= Cx_\delta(t)
\end{aligned}$$

Problem 3.7

From

$$r(t) = \int_{t_0}^{t} v(\sigma)\phi(\sigma)d\sigma$$

taking derivative with respect to time $\dot{r}(t) = v(t)\phi(t)$, and

$$\phi(t) \leq \psi(t) + \int_{t_0}^{t} v(\sigma)\phi(\sigma)d\sigma \quad \Rightarrow \quad \phi(t) \leq \psi(t) + r(t)$$

multiplying by $v(t) \geq 0$

$$\underbrace{\phi(t)v(t)}_{\dot{r}(t)} \leq \psi(t)v(t) + v(t)r(t) \quad \Rightarrow \quad \dot{r}(t) - r(t)v(t) \leq \psi(t)v(t)$$

Multiply both sides by $\exp\left(-\int_{t_0}^{t} v(\tau)d\tau\right)$,

$$\begin{aligned}
\dot{r}(t)e^{-\int_{t_0}^{t} v(\tau)d\tau} - r(t)v(t)e^{-\int_{t_0}^{t} v(\tau)d\tau} &\leq v(t)\psi(t)e^{-\int_{t_0}^{t} v(\tau)d\tau} \\
\frac{d}{dt}\left[r(t)e^{-\int_{t_0}^{t} v(\tau)d\tau}\right] &\leq v(t)\psi(t)e^{-\int_{t_0}^{t} v(\tau)d\tau}
\end{aligned}$$

Integrating both sides

$$r(t)e^{-\int_{t_0}^{t} v(\tau)d\tau} \leq \int_{t_0}^{t} v(\sigma)\psi(\sigma)e^{-\int_{t_0}^{\sigma} v(\tau)d\tau}d\sigma$$

multiplying both sides by $\exp\left(\int_{t_0}^{t} v(\tau)d\tau\right)$

$$\begin{aligned}
r(t) &\leq \left(\int_{t_0}^{t} v(\sigma)\psi(\sigma)e^{\int_{\sigma}^{t_0} v(\tau)d\tau} d\sigma\right) e^{\int_{t_0}^{t} v(\tau)d\tau} \\
r(t) &\leq \int_{t_0}^{t} v(\sigma)\psi(\sigma)e^{\int_{\sigma}^{t} v(\tau)d\tau}
\end{aligned}$$

From $\phi(t) \leq \psi(t) + r(t)$

$$\phi(t) \leq \psi(t) + \int_{t_0}^{t} v(\sigma)\psi(\sigma)e^{\int_{\sigma}^{t} v(\tau)d\tau} d\sigma$$

Problem 3.12

$\dot{x}(t) = A(t)x(t)$, $x(t_0) = x_0$, integrating both sides and taking norms

$$\begin{aligned}
x(t) - x_0 &= \int_{t_0}^{t} A(\tau)x(\tau)d\tau \\
\|x(t)\| &= \left\|\int_{t_0}^{t} A(\tau)x(\tau)d\tau + x_0\right\| \\
\|x(t)\| &\leq \left\|\int_{t_0}^{t} A(\tau)x(\tau)d\tau\right\| + \|x_0\|
\end{aligned}$$

Using Gronwall-Bellman inequality with $\phi(t) = \|x(t)\|$, $\psi(t) = \|x_0\|$, $v(t) = \|A(t)\|$,

$$\|x(t)\| \leq \|x_0\| + \underbrace{\|x_0\| \int_{t_0}^{t} \|A(t)\| \, e^{\int_{\sigma}^{t} \|A(\tau)\| d\tau} d\sigma}_{\text{integrating by parts}}$$

$$\|x(t)\| \leq \|x_0\| - \|x_0\| + \|x_0\| \, e^{\int_{t_0}^{t} \|A(\tau)\| d\tau}$$

so

$$\|x(t)\| \leq \|x_0\| \, e^{\int_{t_0}^{t} \|A(\tau)\| d\tau}$$

EEE 582 HW # 3 SOLUTIONS

Problem 4.6

The unique solution for $\dot{X}(t) = X(t)A(t)$, $X(t_0) = X_0$ is

$$X(t) = \Phi_A(t, t_0)X_0$$

on the other hand, take the transpose system $\dot{X}^\top = A^\top X^\top$, this also has a solution

$$X^\top(t) = \Phi_{A^\top}(t, t_0)X_0^\top \Rightarrow X(t) = X_0\Phi_{A^\top}^\top(t, t_0)$$

For the second part, let $\Phi_1(t, \tau)$, $\Phi_2(t, \tau)$ be the transition matrices for $A_1(t)$ and $A_2(t)$, respectively. Propose as a solution

$$X(t) = \Phi_1(t, t_0)X_0\Phi_2(t, t_0) + \int_{t_0}^{t} \Phi_1(t, \sigma)F(\sigma)\Phi_2(t, \sigma)d\sigma$$

taking $\frac{d}{dt}$

$$\dot{X}(t) = \underbrace{\dot{\Phi}_1(t, t_0)X_0}_{A_1(t)}\underbrace{\Phi_2(t, t_0)}_{X(t)} + \underbrace{\Phi_1(t, t_0)}_{X(t)}\underbrace{X_0\dot{\Phi}_2(t, t_0)}_{A_2^\top(t)} + \underbrace{\Phi_1(t, t)}_{I}F(t)\underbrace{\Phi_2(t, t)}_{I}$$

so

$$\dot{X}(t) = A_1(t)X(t) + X(t)A_2^\top(t) + F(t)$$

To prove uniqueness, pick "two solutions" and assume them different

$$\begin{aligned}\dot{X}_1(t) &= A_1(t)X_1(t) + X_1(t)A_2^\top(t) + F(t)\\ \dot{X}_2(t) &= A_1(t)X_2(t) + X_2(t)A_2^\top(t) + F(t)\end{aligned}$$

produce the difference between the two

$$\dot{Z}(t) = A_1(t)Z(t) + Z(t)A_2^\top(t), \quad Z(t_0) = 0$$

with $Z(t) = X_1(t) - X_2(t)$. Integrating both sides, taking norms and using Gronwall-Bellman lemma we get

$$\|Z(t)\| \leq \|Z_0\| e^{\int_{t_0}^{t}\left\|A_1(\tau)+A_2^\top(\tau)\right\|d\tau} \tag{1}$$

$$\|Z(t)\| \leq 0 \tag{2}$$

the last inequality imply that $Z(t) = 0$ for all t which in turn implies that $X_1(t) = X_2(t)$. Hence there is just one solution.

Problem 4.8

($\Leftarrow$) Assume $A(t)A(\tau) = A(\tau)A(t)\ \forall t, \tau$ then

$$A(t) - A(\tau) - A(\tau)A(t) = 0\ \forall t, \tau$$

integrating both sides

$$\int_{\tau}^{t} \left(A(t)A(\sigma) - A(\sigma)A(t)\right) d\sigma = 0$$

and since the difference is zero for all t, τ

$$A(t)\int_{\tau}^{T} A(\sigma)d\sigma = \left(\int_{\tau}^{t} A(\sigma)d\sigma\right)A(t)$$

($\Rightarrow$) Assume $A(t)\int_{\tau}^{T} A(\sigma)d\sigma = \left(\int_{\tau}^{t} A(\sigma)d\sigma\right)A(t)$, then

$$\int_{\tau}^{t} \left(A(t)A(\sigma) - A(\sigma)A(t)\right) d\sigma = 0$$

suppose $A(t)A(\tau) = A(\tau)A(t)$ is false $\Rightarrow A(t)A(\tau) - A(\tau)A(t) \neq 0$, let $v \in I\!R^n \neq 0$

$$v^\top \left[\int_\tau^t (A(t)A(\sigma) - A(\sigma)A(t))\, d\sigma \right] v = 0$$

and can be written as

$$\int_\tau^t v^\top [A(t)A(\sigma) - A(\sigma)A(t)]\, v d\sigma = 0$$

but $[A(t)A(\sigma) - A(\sigma)A(t)]\, v = f(t,\tau) \forall t, tau$. From the "false assumption and continuity we know that there exists a neighborhood around σ_0 ($|x - \sigma_0| \leq \delta$) for which $f(t,\tau) > \epsilon$. Let $\tau < \sigma_0 < 0$, then

$$\int_\tau^r f(t,\sigma) d\sigma = \underbrace{\int_\tau^{\sigma_0 - \delta} f(t,\sigma) d\sigma}_{=0} + \int_{\sigma_0-\delta}^{\sigma_0+\delta} f(t,\sigma) d\sigma + \underbrace{\int_{\sigma_0+\delta}^t f(t,\sigma) d\sigma}_{=0}$$

and

$$\int_{\sigma_0-\delta}^{\sigma_0+\delta} f(t,\sigma) d\sigma > \int_{\sigma_0-\delta}^{\sigma_0+\delta} \epsilon d\sigma > 2\epsilon\delta \neq 0$$

by contradiction we are done.

Problem 4.13

Using the fact that $\frac{\partial}{\partial t}\Phi_A(t,\tau) = A(t)\Phi_A(t,\tau)$, $\Phi_A(\tau,\tau) = I$,

$$\begin{aligned}
\frac{\partial}{\partial t}\Phi_A(t,\tau) &= \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{bmatrix} \begin{bmatrix} \Phi_{11}(t,\tau) & \Phi_{12}(t,\tau) \\ \Phi_{21}(t,\tau) & \Phi_{22}(t,\tau) \end{bmatrix} \\
\frac{\partial}{\partial t}\Phi_{11}(t,\tau) &= A_{11}(t)\Phi_{11}(t,\tau),\ \Phi_{11}(\tau,\tau) = I \\
\frac{\partial}{\partial t}\Phi_{22}(t,\tau) &= A_{22}(t)\Phi_{22}(t,\tau),\ \Phi_{22}(\tau,\tau) = I \\
\frac{\partial}{\partial t}\Phi_{12}(t,\tau) &= A_{11}(t)\Phi_{12}(t,\tau) + A_{12}(t)\Phi_{22}(t,\tau),\ \Phi_{12}(\tau,\tau) = 0 \\
\frac{\partial}{\partial t}\Phi_{21}(t,\tau) &= A_{22}(t)\Phi_{21}(t,\tau),\ \Phi_{21}(\tau,\tau) = 0 \Rightarrow \Phi_{21}(t,\tau) = 0
\end{aligned}$$

so

$$\Phi_A(t,\tau) = \begin{bmatrix} \Phi_{11}(t,\tau) & \Phi_{12}(t,\tau) \\ 0 & \Phi_{22}(t,\tau) \end{bmatrix}$$

writing the differential equation

$$\dot{\Phi}_{12}(t,\tau) = A_{11}(t)\Phi_{12}(t,\tau) + A_{12}(t)\Phi_{22}(t,\tau)$$

The solution to the homogenous equation is $\Phi_{12}(t,\tau) = \Phi_{11}(t,\tau)$ and the solution to the differential equation is

$$\Phi_{12}(t,\tau) = \Phi_{11}(t,\tau)\underbrace{\Phi_{12}(\tau,\tau)}_{=0} + \int_\tau^t \Phi_{11}(t,\sigma)A_{12}(\sigma)\Phi_{22}(\sigma,\tau) d\sigma$$

then

$$\Phi_{12}(t,\tau) = \int_\tau^t \Phi_{11}(t,\sigma)A_{12}(\sigma)\Phi_{22}(\sigma,\tau) d\sigma$$

Problem 4.25

From the Peano-Baker formula

$$\Phi(t+\sigma,\sigma) = I + \int_\sigma^{t+\sigma)} (\tau) d\tau + \sum_{k=2}^{\infty} \int_\sigma^{t+\sigma} A(\tau_1) \int_\sigma^{\tau_1} A(\tau_2) \cdots \int_\sigma^{\tau_{k-1}} A(\tau_k) d\tau_k \cdots d\tau_1$$

From the exponential matrix series representation

$$e^{\bar{A}_t(\sigma)t} = I + \bar{A}_t(\sigma)t + \sum_{k=2}^{\infty} \frac{1}{k!}\bar{A}_t^k(\sigma)t^k$$

with $\bar{A}_t(\sigma)t = \int_\sigma^T A(\tau)d\tau$. Let $R(t,\sigma) = \Phi(t+\sigma,\sigma) - e^{\bar{A}_t(\sigma)t}$, then $\|R(t,\sigma)\| = \left\|\Phi(t+\sigma,\sigma) - e^{\bar{A}_t(\sigma)t}\right\|$

$$\begin{aligned} \|R(t,\sigma)\| &= \left\| I + \int_\sigma^{t+\sigma)} (\tau)d\tau + \sum_{k=2}^{\infty} \int_\sigma^{t+\sigma} A(\tau_1) \int_\sigma^{\tau_1} A(\tau_2) \cdots \int_\sigma^{\tau_{k-1}} A(\tau_k)d\tau_k \cdots d\tau_1 - \cdots \right. \\ &\quad \left. \cdots \quad -I - \bar{A}_t(\sigma)t + \sum_{k=2}^{\infty} \frac{1}{k!}\bar{A}_t^k(\sigma)t^k \right\| \end{aligned}$$

using the triangle inequality

$$\begin{aligned} \|R(t,\sigma)\| &\leq \sum_{k=2}^{\infty} \int_\sigma^{t+\sigma} \|A(\tau_1)\| \cdots \int_\sigma^{\tau_{k-1}} \|A(\tau_k)\| \, d\tau_k \cdots d\tau_1 + \sum_{k=2}^{\infty} \frac{t^k}{k!} \left\|\bar{A}_t(\sigma)\right\|^k \\ \|R(t,\sigma)\| &\leq \sum_{k=2}^{\infty} \frac{2}{k!}\alpha^k t^k = \alpha^2 t^2 \sum_{k=2}^{\infty} \frac{2}{k!}\alpha^{k-2}t^{k-2} \end{aligned}$$

changing variables and noting that $\frac{2}{k!} \leq \frac{1}{(k-2)!}$, for $k \geq 2$ we get

$$\|R(t,\sigma)\| \leq \alpha^2 t^2 \sum_{k=2}^{\infty} \frac{1}{m!}\alpha^m t^m = \alpha^2 t^2 e^{\alpha t}$$

Problem 5.2a-b

1. The characteristic polynomial is given by $\det(\lambda I - A) = 0$, $\lambda^2 + 2\lambda + 1 = 0$ hence $\lambda_{1,2} = -1$. The exponential matrix is given by: $e^{At} = \beta_0(t)I + \beta_1(t)A$. The functions β_0 and β_1 are given by:

$$\begin{aligned} e^{\lambda t} &= \beta_0 + \lambda\beta_1 \\ te^{\lambda t} &= \beta_1 \quad \text{repeated eigenvalues} \end{aligned}$$

so $\beta_1 = te^{-t}$ and $\beta_0 = e^{-t} + te^{-t}$. Then

$$e^{At} = \begin{bmatrix} t+1 & t \\ -t & 1-t \end{bmatrix} e^{-t}$$

2. The characteristic polynomial is given by $\det(\lambda I - A) = 0$, $(\lambda+1)^3 = 0$ hence $\lambda_{1,2,3} = -1$. The exponential matrix is given by: $e^{At} = \beta_0(t)I + \beta_1(t)A + \beta_2(t)A^2$. The functions β_0, β_1 and β_2 are given by:

$$\begin{aligned} e^{\lambda t} &= \beta_0 + \lambda\beta_1 + \lambda^2\beta_2 \\ te^{\lambda t} &= \beta_1 + 2\beta_2\lambda \quad \text{repeated eigenvalues} \\ t^2 e^{\lambda t} &= 2\beta_2 \end{aligned}$$

so $\beta_2 = \frac{t^2}{2}e^{-t}$, $\beta_1 = te^{-t} + t^2e^{-t}$ and $\beta_0 = e^{-t} + te^{-t} + \frac{t^2}{2}e^{-t}$. Then

$$e^{At} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ te^{-t} & 0 & e^{-t} \end{bmatrix}$$

Problem 5.7

Taking derivatives on both sides

$$\begin{aligned}\frac{d}{dt}\left[A\int_0^t e^{A\sigma}d\sigma\right] &= \frac{d}{dt}\left(e^{At}-I\right)\\ Ae^{At} &= Ae^{At}\end{aligned}$$

Assume initial condition $t=0$ then

$$A\int_0^0 e^{A\sigma}d\sigma = \left(e^{At}-I\right)\big|_{t=0} \Rightarrow 0=0$$

hence the right side is equal to the left side and viceversa.

Assume A^{-1} exists, i.e., $\det(A)\neq 0$, then pre-multiply both sides by A^{-1} and post-multiply both sides by $\left(e^{At}-I\right)^{-1}$

$$\left(\int_0^t e^{A\sigma}d\sigma\right)\left(e^{At}-I\right)^{-1} = A^{-1}$$

from the assumption that A^{-1} exists, it implies that $\left(e^{At}-I\right)\neq 0\ \forall t$ and also that $-\infty<\left(e^{At}-I\right)<\infty\ \forall t$, taking the limit as $t\to\infty$

$$\lim_{t\to\infty}\left[\int_0^t e^{A\sigma}d\sigma\right]\left[\lim_{t\to\infty}\left(e^{At}-I\right)\right] = A^{-1}$$

we need that $\lim_{t\to\infty}\left(e^{At}-I\right)$ be finite this implies that the eigenvalues of A have negative real part.

Under this condition we can write

$$-\int_0^\infty e^{A\sigma}d\sigma = A^{-1} \Rightarrow A^{-1} = \int_\infty^0 e^{A\sigma}d\sigma$$

Problem 5.14

Since $A(t)$ is diagonal then $\Phi_{ii}(t,\tau)=\exp\left(\int_\tau^t a_{ii}(\sigma)d\sigma\right)$, so

$$\Phi(t,\tau) = \left[\begin{array}{cc} e^{-2\sigma+\frac{1}{2}\sin 2\sigma} & 0\\ 0 & e^{-3\sigma+\frac{1}{2}\sin 2\sigma}\end{array}\right]\Bigg|_\tau^t$$

A(t) has period $T=\pi$, then $R=\frac{1}{T}\ln\Phi(T,0)=\left[\begin{array}{cc}-2 & 0\\ 0 & -3\end{array}\right]$ and

$$P(t)=\Phi(t,0)e^{-Rt}=\left[\begin{array}{cc} e^{\frac{1}{2}\sin 2t} & 0\\ 0 & e^{\frac{1}{2}\sin 2t}\end{array}\right]$$

Problem 5.16

From the Floquet decomposition, $\Phi_A(t,\tau)=P(t)e^{R(t-\tau)}P^{-1}(\tau)$ the solution to the differential equation is

$$x(t)=P(t)e^{R(t-\tau)}P^{-1}(\tau)x(\tau)$$

with $x(\tau)$ the initial condition, pre-multiplying both sides of () by $P^{-1}(t)$ and let $z(t)=P^{-1}(t)x(t)$, then we can write () as

$$z(t)=e^{R(t-\tau)}z(\tau)$$

which is the solution to the differential equation

$$\dot{z}(t)=Rz(t)$$

where R is a constant matrix and the change of variables is given by $x(t)=P(t)z(t)$.

Problem 5.17

Since $A(t)$ is T-periodic we can write

$$\Phi(t,t_0) = P^{-1}(t)e^{R(t-t_0)}P(t_0)$$

where $P(t)$ is continuous, T-periodic and invertible at every t. Let $S = P^{-1}(t_0)RP(t_0)$, $Q(t,t_0) = P^{-1}(t)P(t_0)$, then $Q(t,t_0)$ is continuous, T-periodic and invertible at every t, so $Q(t+T,t_0) = Q(t,t_0)$ and $R = P(t_0)SP^{-1}(t_0)$ (notice that $P(t_0)$ is a similarity transformation between R and S

$$\Phi(t,t_0) = P^{-1}(t)e^{R(t-t_0)}P(t_0) = P^{-1}(t)P(t_0)e^{S(t-t_0}P^{-1}(t_0)P(t_0) = Q(t,t_0)e^{S(t-t_0)}$$

EEE 582 HW # 4 SOLUTIONS

Problem 6.3

1. $a(t) = 0$, $A(t)$ is constant and its eigenvalues are 0 and -1, hence the system can not be uniformly exponentially stable.
2. $a(t) = -1$, $A(t)$ is constant and its eigenvalues are -1; since both of them have negative real part, the system is uniformly exponentially stable.
3. $a(t) = -t$

$$x_1(t) = x_{0_1} e^{-\frac{1}{2}(t^2 - t_0^2)} + \int_{t_0}^{t} e^{\int_s^t a(\sigma) d\sigma} x_2(s) ds$$
$$x_2(t) = x_{0_2} e^{-(t-t_0)}$$

The first term of $x_1(t)$ is not boundded uniformly with respect to t_0 (take $t_0 \to -\infty$), therefore the system is not UES.

4. $a(t) = -e^{-t}$

$$\Phi_{11}(t,\tau) = \exp\left(\int_{\tau}^{t} -e^{-s} ds\right) = e^{e^{-t}} e^{-e^{-\tau}}$$

For $\tau = 0$ we get

$$\lim_{t\to\infty} \Phi_{11}(t,0) = \frac{1}{e}$$

This implies that the system is not asymptotically stable.

5. $a(t) = \begin{cases} -1, & t < 0 \\ -e^{-t}, & t \geq 0 \end{cases}$

For $t_0 \geq 0$, this case is as in Exercise 4, and hence the system can not be UES.

Problem 6.7

From $A(t) = -A^\top(t)$, $\forall t \in \mathbb{R}$, $\dot{x}(t) = A(t)x(t) = -A^\top(t)x(t)$. Let $\Phi_A(t,\tau)$ be the state transition matrix of $A(t)$. Then $\Phi_A^\top(\tau, t)$ is the state transition matrix of $-A^\top(t)$ (Property 4.5). So, for any x_0 we have $x(t) = \Phi_A(t,t_0)x_0 = \Phi_A^\top(t_0,t)x_0$. Hence, $\Phi_A(t,t_0) = \Phi_A^\top(t_0,t)$. Multiplying both sides from the left with $\Phi_A^\top(t,t_0)$ we get $\Phi_A(t,t_0)\Phi_A^\top(t,t_0) = [\Phi_A(t,t_0)\Phi_A(t_0,t)]^\top = I$. So, $\Phi_A(t,t_0)$ is uniformly bounded and, in fact, $\|\Phi_A(t,t_0)\| = 1$. This implies that $\dot{x}(t) = A(t)x(t)$ is uniformly stable.

Next, for $P(t)$ to be a Lyapunov transformation we need that

$$\|P(t)\| < \rho_1, \ \left\|P^{-1}(t)\right\| < \rho_2$$

Since $P(t) = \Phi_A(t,0)$, we have $\|P(t)\| = 1$. On the other hand, $P^{-1}(t) = \Phi_A(0,t)$ and, from the previous expression, we have again that $\|P^{-1}(t)\| = 1$. Hence $P(t)$ is a Lyapunov transformation.

Problem 6.8

($\Rightarrow$)

Assume $\dot{x} = A(t)x(t)$ is UES, then $\exists \gamma, \lambda > 0$ such that $\|\Phi_A(t,\tau)\| \leq \gamma e^{-\lambda(t-\tau)}$. The state transition matrix for $\dot{z}(t) = A^\top(-t)z(t)$ is $\Phi_A^\top(-\tau,-t)$. Then

$$\left\|\Phi_A^\top(-\tau,-t)\right\| = \|\Phi_A(-\tau,-t)\| \leq \gamma e^{-\lambda(-\tau-(-t))} \Rightarrow$$
$$\left\|\Phi_A^\top(-\tau,-t)\right\| \leq \gamma e^{-\lambda(t-\tau)}$$

and this implies that the linear state equation $\dot{z}(t) = A^\top(-t)z(t)$ is UES.

($\Leftarrow$)

Assume $\dot{z}(t) = A^\top(-t)$ is UES, then $\exists\gamma, \lambda > 0$ such that $\left\|\Phi_{A^\top(-t)}(t,\tau)\right\| \leq \gamma e^{\lambda(t-\tau)}$. But the state transition matrix for $\dot{x}(t) = A(t)x(t)$ is $\Phi^\top_{A^\top(-t)}(-\tau,-t)$. Then

$$\left\|\Phi^\top_{A^\top(-t)}(-\tau,-t)\right\| = \left\|\Phi_{A^\top(-t)}(-\tau,-t)\right\| \leq \gamma e^{\lambda(-\tau-(-t))}$$
$$\left\|\Phi_{A^\top(-t)}(-\tau,-t)\right\| \leq \gamma e^{\lambda(t-\tau)}$$

and this implies that the linear state equation $\dot{x}(t) = A(t)x(t)$ is UES.

Problem 6.11

We know that

$$\|x(t)\| = \|x_0\| \exp\left(\frac{1}{2}\int \lambda_{\max}\left(A + A^\top\right)\right)$$

so if $A + A^\top < 0$ implies that $\Re\{\lambda_i(A)\} < 0$ for $i = 1, ..., n$, —n being the dimension of the matrix A— and the system is UES.

Since $F = F^\top > 0$ we can factorize $F = F^{\frac{1}{2}}F^{\frac{1}{2}}$ such that $F^{\frac{1}{2}} = F^{\frac{1}{2}^\top} > 0$, let $z = F^{-\frac{1}{2}}x$, then $x = F^{\frac{1}{2}}z$ and the differential equation becomes

$$F^{\frac{1}{2}}\dot{z} = FAF^{\frac{1}{2}}z$$
$$\dot{z} = F^{\frac{1}{2}}AF^{\frac{1}{2}}z$$

Next,

$$F^{\frac{1}{2}}AF^{\frac{1}{2}} + F^{\frac{1}{2}}A^\top F^{\frac{1}{2}} = F^{\frac{1}{2}}\left(A + A^\top\right)F^{\frac{1}{2}} = F^{\frac{1}{2}}\underbrace{\left(A + A^\top\right)}_{<0}\left(F^{\frac{1}{2}}\right)^\top < 0$$

Therefore $\Re\{\lambda_i(FA)\} < 0$ and the system is UES.

Problem 6.13

$\dot{x}(t) = A(t)x(t)$ US implies that $\exists\gamma > 0$ such that $\|\Phi(t,\tau)\| \leq \gamma$.

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t}\Phi(t,\tau)f(\tau)d\tau$$
$$\|x(t)\| \leq \|\Phi(t,t_0)\|\,\|x_0\| + int_{t_0}^{t}\,\|\Phi(t,\tau)\|\,\|f(\tau)\|\,d\tau$$
$$\|x(t)\| \leq \gamma\,\|x_0\| + \gamma\int_{t_0}^{t}\|f(\tau)\|\,d\tau$$

Thus, if

$$\lim_{t\to\infty}\int_{t_0}^{t}\|f(t)\|\,dt \leq \eta < \infty$$

then $x(t)$ is bounded.

In general, (arbitrary A, f) this condition is also necessary: Let $A(t)$ be a constant matrix equal to zero and $f(t)$ having the same sign for all t. Then if this condition is violated, $x(t)$ is unbounded.

Problem 7.1

For US wee need

$$\eta I \leq Q(t) \leq \rho I$$
$$A^\top(t)Q(t) + Q(t)A(t) + \dot{Q}(t) \leq 0$$

Pick $Q(t) = I$, then $\eta I \leq Q(t) \leq \rho I$ with $\eta = \rho = 1$, and

$$\underbrace{A^\top(t) + A(t)}_{=0} + 0 \leq 0$$
$$0 \leq 0$$

There is not a $Q(t)$ that results in UES. As a counter-example, pick $A(t) = 0$ that satisfies the hypothesis but is not UES.

Problem 7.8

For US the set of conditions is derived from

$$\alpha I \leq Q(t) \leq \beta I$$
$$A^\top(t)Q(t) + Q(t)A(t) + \dot{Q}(t) \leq 0$$

which are

$$\alpha \leq a_1(t) \leq \beta$$
$$\dot{a}_1(t) \leq 0$$
$$0 \leq a_2(t)$$

For UES the set of conditions is derived from

$$\alpha I \leq Q(t) \leq \beta I$$
$$A^\top(t)Q(t) + Q(t)A(t) + \dot{Q}(t) \leq -\nu I$$

which are

$$\alpha \leq a_1(t) \leq \beta$$
$$\dot{a}_1(t) \leq -\nu$$
$$\frac{\nu}{2} \leq a_2(t)$$

Problem 7.9 For UES the set of conditions is derived from

$$\alpha I \leq Q(t) \leq \beta I$$
$$A^\top(t)Q(t) + Q(t)A(t) + \dot{Q}(t) \leq -\nu I$$

which are

$$0 < \alpha < \frac{1}{\sqrt{2}}$$
$$\beta = \frac{2\alpha + 1}{\alpha}$$
$$\alpha \leq a(t) \leq \frac{1}{2\alpha}$$
$$\nu a^2(t) - 2a^3(t) \leq \dot{a}(t) \leq a(t) - \frac{\nu}{2}$$

Problem 7.11

We can write the equation as

$$\left(A^\top + \mu I\right) Q + Q \left(A + \mu I\right) = -M$$

By Theorem 7.11 we conclude that all eigenvalues of $A + \mu I$ have negative real part, that is, if

$$0 = \det\left(\lambda I - (A + \mu I)\right) = \det\left(\left(\lambda - \mu\right) I - A\right)$$

then $\Re[\lambda] < 0$. Since $\mu > 0 \Rightarrow \Re[\lambda - \mu] < -\mu$, that is, all the eigenvalues of A have real parts strictly less than $-\mu$.

Suppose all eigenvalues of A have real parts strictly less than $-\mu$. Then, as above, all eigenvalues of $A + \mu I$ have negative real part. Then, by Theorem 7.1, given a symmetric positive definite matrix M, there exists a unique, symmetric, positive definite matrix Q such that $\left(A^\top + \mu I\right) Q + Q(A + \mu I) = -M$ holds. This implies that $A^\top Q + QA + 2\mu Q = -M$ holds.

Problem 7.12

Substitute Q in $A^\top Q + QA = -M$ such that

$$A^\top e^{A^\top t} Q e^{At} + e^{A^\top t} Q e^{At} A + \underbrace{A^\top \int_0^t e^{A^\top \sigma} M e^{A\sigma} d\sigma + \left[\int_0^t e^{A^\top \sigma} M e^{A\sigma} d\sigma\right] A}_{e^{A^\top \sigma} M e^{A\sigma}} = -M$$

$$A^\top e^{A^\top t} Q e^{At} + e^{A^\top t} Q e^{At} A + e^{A^\top t} M e^{At} - M = -M$$

$$A^\top e^{A^\top t} Q e^{At} + e^{A^\top t} Q e^{At} A + e^{A^\top t}\left(-A^\top Q - QA\right) e^{At} = 0$$

$$\underbrace{\left[A^\top e^{A^\top t} - e^{A^\top t} A\right]}_{=0} Q e^{At} + e^{A^\top t} Q \underbrace{\left[e^{At} A - A e^{At}\right]}_{=0} = 0$$

$$0 = 0$$

Problem 7.16

For an arbitrary but fixed $t \geq 0$, let x_a be such that

$$\|x_a\| = 1, \ \left\|e^{At} x_a\right\| = \left\|e^{At}\right\|$$

By Theorem 7.11 the unique solution of $QA + A^\top Q = -M$ is the symmetric, positive definite matrix

$$Q = \int_0^\infty e^{A^\top \sigma} M e^{A\sigma} d\sigma$$

Then we can write

$$\int_t^\infty x_a^\top e^{A^\top \sigma} M e^{A\sigma} x_a d\sigma \leq \int_0^\infty x_a^\top e^{A^\top \sigma} M e^{A\sigma} x_a d\sigma = x_a^\top Q x_a \leq \lambda_{\max}(Q) = \|Q\|$$

Making a change of integration variable from σ to $\tau = \sigma - t$,

$$\int_t^\infty x_a^\top e^{A^\top \sigma} M e^{A\sigma} x_a d\sigma = \int_0^\infty x_a^\top e^{A^\top (t+\tau)} M e^{A(t+\tau)} x_a d\tau = x_a^\top e^{A^\top t} Q e^{At} x_a \geq \lambda_{\min}(Q) \left\|e^{At} x_a\right\|^2 = \frac{\left\|e^{At}\right\|^2}{\|Q^{-1}\|}$$

Hence,

$$\frac{\left\|e^{At}\right\|^2}{\|Q^{-1}\|} \leq \|Q\|$$

and, since t is arbitrary,

$$\max_{t \geq 0} \left\|e^{At}\right\| \leq \sqrt{\|Q\| \left\|Q^{-1}\right\|}$$

EEE 582 HW # 5 SOLUTIONS

Problem 8.6

The solution to the differential equation for any x_0, $t_0 \geq 0$ is

$$x(t) = \Phi_{A+F}(t,t_0)x_0 = \Phi_A(t,t_0)x_0 + \int_{T_0}^{t} \Phi_A(t,\sigma)F(\sigma)x(\sigma)d\sigma$$

since $\dot{x}(t) = A(t)x(t)$ is UES, there are $\gamma > 0$, $\lambda > 0$ such that

$$\|x(t)\| \leq \gamma e^{-\lambda(t-t_0)}\|x_0\| + \int_{t_0}^{t} \gamma e^{-\lambda(t-\sigma)}\|F(\sigma)\|\|x(\sigma)\| d\sigma$$

$$e^{\lambda T}\|x(t)\| \leq \gamma e^{\lambda t_0}\|x_0\| + \int_{t_0}^{t} \gamma \|F(\sigma)\| e^{\lambda\sigma}\|x(\sigma)\| d\sigma$$

Using Gronwall-Bellman lemma

$$e^{\lambda t}\|x(t)\| \leq \gamma e^{\lambda t}\|x_0\| \exp\left(\int_{t_0}^{t} \gamma \|F(\sigma)\| d\sigma\right)$$

$$\|x(t)\| \leq \gamma e^{-\lambda(t-t_0)} e^{\gamma\beta}\|x_0\|$$

then

$$\|x(t)\| \leq \gamma_1 e^{-\lambda(t-t_0)}\|x_0\|$$

with $\gamma_1 = \gamma e^{\gamma\beta}$

Problem 8.7

Since $F(t)$ is continuous, we can partition the interval $[t_0, t]$ such that $t_0 < t_1 < t$ and $\|F(\sigma)\| < \frac{\epsilon}{\gamma}$ for $\sigma > t_1$ and $\|F(\sigma)\| < \beta$ for $t_0 < \sigma < t_1$. Then

$$\|x(t)\| \leq \|\Phi_A(t,t_0)\|\|x_0\| + \int_{t_0}^{t_1} \|\Phi_A(t,\sigma)\|\|F(\sigma)\|\|x(\sigma)\| d\sigma + \int_{t_1}^{t} \|\Phi_A(t,\sigma)\|\|F(\sigma)\|\|x(\sigma)\| d\sigma$$

$$\|x(t)\| \leq \gamma e^{-\lambda(t-t_0)}\|x_0\| + \int_{t_0}^{t_1} \gamma\beta e^{-\lambda(t-\sigma)}\|x(\sigma)\| d\sigma + \int_{t_1}^{t} \gamma\epsilon e^{-\lambda(t-\sigma)}\|x(\sigma)\| d\sigma$$

and using Exercise 8.6

$$\|x(t)\| \, \gamma e^{-\lambda(t-t_0)} e^{\gamma(\beta+\epsilon)}$$

Hence, $\lim_{t\to\infty} \|x(t)\| = 0 \Rightarrow x(t) \to 0$.

Problem 8.8

Using Theorem 8.7 it follows that the solution of $A^\top(t)Q(t) + Q(t)A(t) = -I$ is $Q(t) = \int_0^\infty e^{A^\top(t)\sigma}e^{A(t)\sigma}d\sigma$ which is continously-differentiable and satisfies $\eta I \leq Q(t) \leq \rho I$, for all t, where η and ρ are positive constants. Then with $F(t) = A(t) - \frac{1}{2}Q^{-1}(t)\dot{Q}(t)$

$$F^\top(t)Q(t) + Q(t)F(t) + \dot{Q}(t) = A^\top(t)Q(t) + Q(t)A(t) = -I$$

Thus, using Theorem 7.4, $\dot{x}(t) = F(t)x(t)$ is UES.

Problem 9.1

The controllability matrix is

$$W_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & \alpha+2 & 2\alpha+2 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

For controllability we need that $\det(W_c) \neq 0$. But $\det(W_c) = -\alpha$, hence the system is controllable for all $\alpha \neq 0$.

The observability matrix is

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & \alpha & 0 \\ 0 & 2 & 0 \\ 0 & \alpha & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

notice that $\text{rank}(W_o) = 2 < 3$ for all $\alpha \in I\!R$, hence the system is not observable for any α

Problem 9.4

$$W(t, t_f) = \int_t^{t_f} \Phi(t,\tau)B(\tau)B^\top(\tau)\Phi^\top(t,\tau)d\tau$$

$$W(t_f, t_f) = 0$$

$$\frac{d}{dt}W(t,t_f) = -\int_{t_f}^{t}\left(\frac{d}{dt}\Phi(t,\tau)\right)B(\tau)B^\top(\tau)\Phi^\top(t,\tau)d\tau - \int_{t_f}^{t}\Phi(t,\tau)B(\tau)B\top(\tau)\left(\frac{d}{dt}\Phi\top(t,\tau)\right)d\tau - B(t)B^\top(t)$$

$$\frac{d}{dt}W(t,t_f) = A(t)W(t,t_f) + W(t,t_f)A^\top(t) - B(t)B^\top(t)$$

Using $\dot{P}^{-1}(t) = -P^{-1}(t))\dot{P}(t)P^{-1}(t)$, we can write

$$\frac{d}{dt}W^{-1}(t,t_f) = -W^{-1}(t,t_f)A(t) - A^\top(t)W^{-1}(t,t_f) + W^{-1}(t,t_f)B(t)B^\top(t)W^{-1}(t,t_f)$$

$$W(t_o, t_f) = \int_{t_0}^{t_f} \Phi(t_0,\tau)B(\tau)B^\top(\tau)\Phi^\top(t_o,\tau)d\tau$$

Then, for $t_0 < t < t_f$

$$W(t_o,t_f) = \int_{t_0}^{t}\Phi(t_0,\tau)B(\tau)B\top(\tau)d\tau + \int_{t}^{t_f}\Phi(t_0,\tau)B(\tau)B^\top(\tau)d\tau$$

$$W(t_o,t_f) = W(t_0,t) + \Phi(t_0,t)W(t,t_f)\Phi^\top(t_0,t)$$

Problem 9.7

($\Rightarrow$) $[A,B]$ controllable $\Leftrightarrow W_{c_A} = \int_{t_0}^{t_f} \Phi_A(t,\tau)BB^\top\Phi_A^\top(t,\tau)d\tau > 0$, but $\Phi_A(t,\tau) = e^{A(t-\tau)}$. Since $(A)(\beta I) = (\beta I)(A) \Rightarrow \Phi_{(A-\beta I)}(t,\tau) = e^{(A-\beta I)(t-t_0)}$ then

$$W_{c_{(A-\beta I)}} = \int_{t_0}^{t_f} e^{(A-\beta I)(t-t_0)}BB^\top e^{(A^\top-\beta I)(t-t_0)}$$

$$W_{c_{(A-\beta I)}} = \int_{t_0}^{t_f} e^{-2\beta(t-\tau)}e^{A(t-\tau)}BB^\top e^{A^\top(t-\tau)}d\tau$$

The function $e^{-2\beta(t-\tau)}$ is bounded above and below for any $\tau \in [t_0, t_f]$ by $0 < \gamma \leq e^{-2\beta(t-\tau)} \leq \delta < \infty$; then

$$0 < \gamma W_{c_A} \leq W_{c_{(A-\beta I)}} \leq \delta W_{c_A} < \infty$$

so $W_{c_{(A-\beta I)}} > 0 \Leftrightarrow [(A-\beta I), B]$ controllable.

($\Leftarrow$) $[(A-\beta I), B]$ controllable $\Leftrightarrow W_{c_{(A-\beta I)}} = \int_{t_0}^{t_f} e^{(A-\beta I)(t-t_0)} BB^\top e^{(A^\top - \beta I)(t-t_0)} d\tau > 0$, and

$$W_{c_A} = \int_{t_0}^{t_f} e^{A(t-\tau)} BB^\top e^{A^\top (t-\tau)} d\tau$$
$$W_{c_A} = \int_{t_0}^{t_f} e^{2\beta(t-\tau)} e^{A(t-\tau)} BB^\top e^{A^\top (t-\tau)} d\tau$$

The function $e^{2\beta(t-\tau)}$ is bounded above and below for any $\tau \in [t_0, t_f]$ such that $0 < \kappa \leq e^{2\beta(t-\tau)} \leq \lambda < \infty$; then

$$0 < \kappa W_{c_{(A-\beta I)}} \leq W_{c_A} \leq \lambda W_{c_{(A-\beta I)}} < \infty$$

so $W_{c_A} > 0 \Leftrightarrow [A, B]$ controllable.

Problem 9.13

From the PBH test, a system is controllable if and only if

$$\left.\begin{aligned} p^\top A &= p^\top \lambda \\ p^\top B &= 0 \end{aligned}\right\} \Rightarrow p^\top \equiv 0$$

then the problem is equivalent to show that $\mathcal{N}(B^\top) = \mathcal{N}(BB^\top)$

($\Rightarrow$) Let (A,B) be controllable then $p^\top B = 0 \Rightarrow p \equiv 0$, for $(A, BB^\top)$, $p^\top BB^\top = 0$ then $\mathcal{N}(B^\top) \subseteq \mathcal{N}(BB^\top)$.

($\Leftarrow$) By contradiction, let $p \in \mathcal{N}(BB^\top) \Rightarrow BB^\top p = 0$, assume $B^\top p \neq 0$ and let $m = B^\top p$, then $m^\top m = p^\top BB^\top p \neq 0$, which the contradiction, so $\mathcal{N}(BB^\top) \subseteq \mathcal{N}(B^\top)$.

The two set inclusions imply $\mathcal{N}(B^\top) = \mathcal{N}(BB^\top)$.

Notice that this property is a fundamental one and is not limited to the time-invariant case (the use of PBH is TI-specific). More generally, we can use the definition to show this equivalence:

- Suppose $\dot{x} = Ax + BB^\top u$ is controllable and let u_* be the input that transfers the arbitrary initial state x_0 to the origin. Then the input $v_* = B^\top u_*$ applied to the system $\dot{x} = Ax + Bv$ also transfers the same initial state to the origin. Since this argument holds for any initial state, $\dot{x} = Ax + Bv$ is controllable.

- Suppose $\dot{x} = Ax + Bu$ is controllable and let u_* be the input that transfers the arbitrary initial state x_0 to the origin. Then the input $v_* : BB^\top v_* = Bu_*$ applied to the system $\dot{x} = Ax + BB^\top v$ also transfers the same initial state to the origin. This input always exists since $\mathcal{R}(B) = \mathcal{R}(BB^\top)$, *even if* $BB^\top$ *is not invertible* (a variation of the proof given above, e.g., using null-range properties). Since this argument holds for any initial state, $\dot{x} = Ax + BB^\top v$ is controllable.

EEE 582 HW # 6 SOLUTIONS

Problem 10.3

- Controllable & observable:

$$\left.\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}\right\} \quad \begin{aligned} Q_c &= \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \\ Q_o &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

- Controllable & not observable

$$\left.\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x \end{aligned}\right\} \quad \begin{aligned} Q_c &= \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ 1 & -3 & 9 \end{bmatrix}, \quad \det(Q_c) = -4 \\ Q_o &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 0 \end{bmatrix}, \quad \det(Q_o) = 0 \end{aligned}$$

- Not controllable & observable

$$\left.\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x \end{aligned}\right\} \quad \begin{aligned} Q_c &= \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \quad \det(Q_c) = 0 \\ Q_o &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ -1 & -2 & 9 \end{bmatrix}, \quad \det(Q_o) = 4 \end{aligned}$$

- Not controllable & not observable

$$\left.\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x \end{aligned}\right\} \quad \begin{aligned} Q_c &= \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \quad \det(Q_c) = 0 \\ Q_o &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -2 & 0 \end{bmatrix}, \quad \det(Q_o) = 0 \end{aligned}$$

Problem 11.3

Let $P(t)$ be a change of variables such that

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned}$$

is equivalent to

$$\begin{aligned} \dot{z}(t) &= \bar{A}(t)z(t) + \bar{B}(t)u(t) \\ y(t) &= \bar{C}(t)z(t) \end{aligned}$$

with

$$\begin{aligned} \bar{A}(t) &= P^{-1}(t)\left(P(t)A(t) + \dot{P}(t)\right) \\ \bar{B}(t) &= P^{-1}(t)B(t) \\ \bar{C}(t) &= C(t)P(t) \\ x(t) &= P(t)z(t) \\ \Phi_{\bar{A}}(t,\tau) &= P(t)\Phi_A(t,\tau)P^{-1}(\tau) \end{aligned}$$

We know that for instantaneous controllability

$$\begin{aligned}Q_c &= \begin{bmatrix} K_0(t) & K_1(t) & \cdots K_{n-1}(t) \end{bmatrix}, \quad \operatorname{rank}(Q_c) = n\\ K_0(t) &= B(t)\\ K_j(t) &= -A(t)K_{j-1}(t) + \dot{K}_{j-1}(t), j = 1, ..., n\end{aligned}$$

For the transformed system

$$\begin{aligned}\bar{Q}_c &= \begin{bmatrix} \bar{K}_0(t) & \bar{K}_1(t) & \cdots \bar{K}_{n-1}(t) \end{bmatrix}, \quad \operatorname{rank}(\bar{Q}_c) = n\\ \bar{K}_0(t) &= \bar{B}(t)\\ \bar{K}_j(t) &= -\bar{A}(t)\bar{K}_{j-1}(t) + \dot{\bar{K}}_{j-1}(t), j = 1, ..., n\end{aligned}$$

but

$$\begin{aligned}\bar{K}_0 &= P(t)B(t) = P(t)K_0\\ \bar{K}_1 &= -\bar{A}\bar{K}_0 + \dot{\bar{K}}_0(t) = P(t)K_1(t)\\ \bar{K}_j(t) &= -\bar{A}(t)\bar{K}_{j-1}(t) + \dot{\bar{K}}_{j-1}(t) = P(t)K_j(t),\end{aligned}$$

Hence $\bar{Q}_c = P(t)Q_c$, and since $P(t)$ is invertible it follows that $\operatorname{rank}(\bar{Q}_c) = n$

Similarly for the observability case.

Problem 12.5

$$\begin{aligned}&\Phi(t,\delta) = 1 \Rightarrow\\ W(t-\delta,t) &= \int_{t-\delta}^{t} \tau^2 e^{-2\tau} d\tau \geq \frac{1}{3}e^{-2t}\left(t^3 - (t-\delta)^3\right)\\ &\geq \frac{1}{3}e^{-2t}\delta\left[\left(\sqrt{3}t - \frac{\sqrt{3}}{2}\delta\right)^2 + \frac{\delta^2}{4}\right] > 0 \quad \forall t\end{aligned}$$

On the other hand, for any $\delta > 0$, $\lim_{t\to\infty} W(t-\delta,t) = 0$ and, consequently, there does not exist an $\epsilon > 0$ such that $W(t-\delta,t) > 0$ for all t.

Problem 13.11

For the time invariant case

$$p^\top A = p^\top \lambda, \quad p^\top B = 0 \Rightarrow p = 0$$

Then

$$p^\top (A + BK) = p^\top \lambda, \quad p^\top B = 0 \Rightarrow p = 0$$

and, therefore, controllability of the open-loop state equation implies controllability of the closed-loop state equation.

In the time-varying case, suppose the open-loop state equation is controllable on $[t_0, t_f]$. Then given $x(t_0) = x_a$, $\exists u_a(t)$ such that the solution $x_a(t)|_{t_f} = 0$. Next, the closed-loop equation

$$\dot{z}(t) = [A(t) + B(t)K(t)]\, z(t) + B(t)v(t)$$

with initial state $z(t_0) = x_a$ and input $v_a(t) = u_a(t) - K(t)x_a(t)$ has the solution $z(t) = x_a(t)$. Thus, $z(t_f) = 0$. Since the argument is valid for any x_0, the closed-loop state equation is controllable on $[t_0, t_f]$.

(See also Pr. 9.13 in HW 5.)

Problem 14.7

Using the hint,

$$K = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} p(A)$$

Also, for the controller canonical form

$$\det(\lambda I - A - bK) = \lambda^n + (a_{n-1} + k_{n-1})\lambda^{n-1} + \cdots + (a_0 + k_0)$$

So, given $p(\lambda) \Rightarrow k = \begin{bmatrix} -c_0 + a_0 & -c_1 + a_1 & \cdots & -c_{n-1} + a_{n-1} \end{bmatrix}$, and $p(A) = A^n + c_{n-1}A^{n-1} + \cdots c_1 A + c_0 I$, then

$$\begin{aligned}
-\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} c_0 I &= \begin{bmatrix} -c_0 & 0 & \cdots & 0 \end{bmatrix} \\
-\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} c_1 A &= \begin{bmatrix} 0 & -c_1 & \cdots & 0 \end{bmatrix} \\
&\vdots \\
-\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} c_{n-1} A^{n-1} &= \begin{bmatrix} 0 & 0 & \cdots & -c_{n-1} \end{bmatrix} \\
-\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} A^n &= \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \end{bmatrix}
\end{aligned}$$

so $k = -\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} p(A) = \begin{bmatrix} a_0 - c_0 & a_1 - c_1 & \cdots & a_{n-1} - c_n - 1 \end{bmatrix}$

For the general case, using a similarity transformation ($z = Tx$), it is possible to express the system $\dot{x} = Ax + bu$ in controllable canonical form $z = A_c z + b_c u$, with $A_c = TAT^{-1}$, $b_c = Tb$, $k_c = kT^{-1}$ $Q_{c_z} = TQ_{c_x}$ and $T = Q_{c_z} Q_{c_x}^{-1}$, $Q_{c_x} = \begin{bmatrix} b & Ab & \cdots A^{n-1}b \end{bmatrix}$, $Q_{c_z} = \begin{bmatrix} b_c & A_c b_c & \cdots A_c^{n-1} b_c \end{bmatrix}$ then

$$\begin{aligned}
k &= k_c T = -\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} p(A_c) T = -\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} T p(A) T^{-1} T \\
k &= -\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} Q_{c_z} Q_{c_x}^{-1} p(A)
\end{aligned}$$

but $Q_{c_z} = \begin{bmatrix} \bar{0} & & 1 \\ & \nearrow & \\ 1 & & * \end{bmatrix}$ hence

$$k = -\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} Q_{c_x}^{-1} p(A) = -\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} b & Ab & \cdots A^{n-1}b \end{bmatrix}^{-1} p(A)$$

Problem 15.2

$$\begin{aligned}
v(t) &= Cx(t) + CLz(t) \\
u(t) &= Mz(t) + NCx(t) + NCLz(t)
\end{aligned}$$

$$\begin{cases} \dot{x}(t) = (A + BNC)x(t) + (BM + BNCL)z(t) \\ \dot{z}(t) = GCx(t) + (F + GCL)z(t) \end{cases}$$

Multiply $\dot{z}(t)$ by L, add to $\dot{x}(t)$ and simplify

$$\dot{x}(t) + L\dot{z}(t) = (A - HC)(x(t) + Lz(t))$$

Let $w(t) = x(t) + Lz(t)$; then the closed-loop system is

$$\begin{bmatrix} \dot{w}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A - HC & 0 \\ GC & F \end{bmatrix} \begin{bmatrix} w(t) \\ z(t) \end{bmatrix}$$

The eigenvalues of the system are given by the eigenvalues of the diagonal elements.

EEE 482 HW#4

Problem 1.
Show that controllability is invariant under similarity transformations and under state feedback.

The first of these basic properties states that if $[A_1, B_1]$ and $[A_2, B_2]$ are related by a similarity transformation then controllability of one is equivalent to controllability of the other.

The second property concerns a system $[A, B]$ to which the feedback $u = Kx + v$ is applied. The new system, from v to x, is $[A+BK, B]$. Again controllability of one is equivalent to controllability of the other.

Problem 2.
Consider the system with transfer function $G(s) = 1/(10s+1)(0.2s+1)$. We would like to design a state-feedback-plus-observer type controller to achieve closed-loop bandwidth around 5 (e.g., closed-loop poles with magnitude 5).

1. Design the controller using pole-placement techniques to compute the state-feedback and observer gains.
2. Design the controller using linear quadratic regulator techniques to compute the state feedback and observer gains. (You will need to try different weights to achieve the desired bandwidth.)
3. Use integrator augmentation to achieve integral action and repeat the designs 1&2.

Problem 3.
In the control of practical systems, the ubiquitous nonlinearities translate in an output offset that depends on the operating conditions (justify this from the linearization of a nonlinear system). To account for such an offset, one may design an observer with integral action. For example, suppose that the state equations are

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du + v$$

where v is the offset (constant for a fixed operating point). Design an observer that estimates both the states x and the offset v. Are the required conditions for observability satisfied?

Problem 4.
In designing an output feedback controller with integral action, it is necessary that the plant has no zeros at the origin. Otherwise, the controller integrator will cancel the plant zero and cause internal stability problems. Consider the SISO system (A,b,c,0) for which the control input is defined as follows

$$\dot{z} = r - y$$

$$u = kx + k_z z$$

Assume that the state x is available for measurement. Show that if (A,b) is c.c. and $c(sI-A)^{-1}b$ has no zero at s=0, then all the eigenvalues of the closed-loop system can be assigned arbitrarily. (For simplicity, assume that A is nonsingular.)

Hint: You need to show that the augmented system is c.c., that is

$$\left[\begin{pmatrix} A & 0 \\ -c & 0 \end{pmatrix}, \begin{pmatrix} b \\ 0 \end{pmatrix}\right]$$

is a c.c. pair. For this, use the PBH test. Notice that the condition that $c(sI-A)^{-1}b$ has no zero at the origin means that $cA^{-1}b$ is non-zero.

EEE 582 LINEAR SYSTEMS HW#1 SOLUTIONS

2.10 Take Laplace transform of both sides, assuming zero I.C. $\Rightarrow$

$$(s^2+2s-3)\,y(s) = (s-1)\,u(s)$$

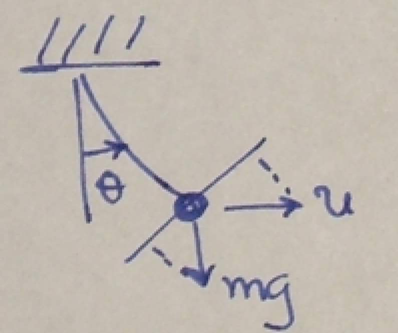
$$\Rightarrow \frac{y(s)}{u(s)} = \frac{(s-1)}{(s+3)(s-1)} = \frac{1}{s+3}$$

$\therefore$ Impulse response is

$$y_{IR}(t) = \mathcal{L}^{-1}\{ y(s)\big|_{\substack{u(t)=\delta(t)\\ u(s)=1}} \}$$

$$= \mathcal{L}^{-1}\{\tfrac{1}{s+3}\} = e^{-3t} \quad t\geq 0$$

2.15 a.



Newton's law in the tangential direction:

$$u\cos\theta - mg\sin\theta = ml\ddot{\theta}$$

Define $x_1 = \theta$, $x_2 = \dot{\theta}$. Then,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\tfrac{g}{l}\sin x_1 - \frac{\cos x_1}{ml}\,u$$

This is a nonlinear system. When θ is small,

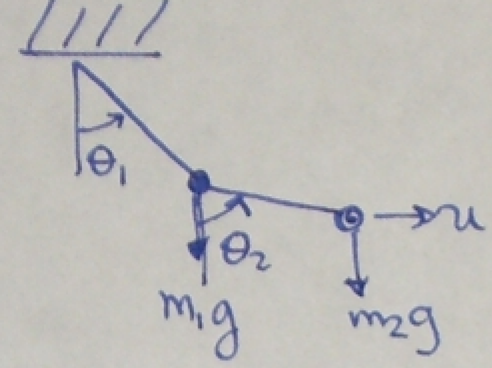
$$\sin x_1 \simeq x_1, \quad \cos x_1 \simeq 1$$

$$\Rightarrow \begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -g/l & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ -1/ml\end{pmatrix}u$$

which is the linearized system around the equilibrium

$$\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}\theta\\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix};\ u = 0$$

2.15 b.



In this case, Newton's law yields:

$$u\cos\theta_2 - m_2 g\sin\theta_2 = m_2 l_2 \ddot{\theta}_2$$

The link tension is: $T = m_2 g\cos\theta_2 + u\sin\theta_2$

which generates torque at m_1, together with $m_1 g$:

$$T\sin(\theta_2-\theta_1) - m_1 g\sin\theta_1 = m_1 l_1 \ddot{\theta}_1$$

Now, define $x = \begin{pmatrix}\theta_1\\ \dot{\theta}_1\\ \theta_2\\ \dot{\theta}_2\end{pmatrix}$ and write the state diff. eqn

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -g/l_1 \sin x_1 + \frac{m_2 g}{m_1 l_1}\cos x_3 \sin(x_3-x_1) + 1/m_1 l_1 \sin x_3 \sin(x_3-x_1)\,u$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -g/l_2 \sin x_3 + \frac{\cos x_3}{m_2 l_2}u$$

This is a nonlinear system. When θ_1, θ_2 are small,

$\sin x_1 \simeq x_1$, $\sin(x_3-x_1) \simeq x_3-x_1$, $\cos x_3 \simeq 1$

$\sin x_3 \simeq x_3$

yielding:

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{g(m_1+m_2)}{m_1 l_1} & 0 & \frac{m_2 g}{m_1 l_1} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -g/l_2 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2 l_2}\end{pmatrix} u$$

This is the linearized system around the equilibrium

$x = 0$, $u = 0$ (where $\theta_1 = \theta_2 = \dot{\theta}_1 = \dot{\theta}_2 = 0$)

2.17 $m\ddot{y} = -k\dot{m} - mg$

$x_1 = y, \quad x_2 = \dot{y}, \quad x_3 = m, \quad u = \dot{m}$

$$\Rightarrow \quad \dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{x_3}u - g$$

$$\dot{x}_3 = u$$

This is a nonlinear system.

Notice that the nonlinearity enters through the control matrix G in the general description

$$\dot{x} = F(x) + G(x)u$$

and it is associated with the changing mass of the lunar module. That changes the system inertia and, therefore, its acceleration characteristics.
For this problem, different linearizations would be used for the different stages of descent.

EEE 582 HW 2 SOLUTIONS

3.4 A = symmetric $n\times m$, $n\geq m$, orthonormal columns.
Then $A^TA = I$. For AA^T, if $n=m$, then $A^T=A^{-1}$ and $AA^T=I$. If $n<m$ nothing can be said in general.

3.7 $x=\begin{pmatrix}1\\1\end{pmatrix}$ is a solution. It is unique because $\mathcal{N}(A)=\{0\}$
(One way to compute it is by the LS formula $(A^TA)^{-1}A^Ty$; this formula will yield the solution, if one exists, or the LS minimizer ~~of~~ the solution does not exist).

When $y=\begin{bmatrix}1\\1\\1\end{bmatrix}$, $rk(A)=2\neq rk([A|y])=3$ and a solution does not exist.
(Using the LS formula one can simply verify $A(A^TA)^{-1}A^Ty-y$ is not zero).

3.12 $\Delta(s)\doteq \det(sI-A)^{-1} = (s-2)^3(s-1) = s^4-7s^3+18s^2-20s+8$

By Cayley Hamilton, $A^4=-8I+20A-18A^2-7A^3$

$\therefore\ Ab=[b, Ab, A^2b, A^3b]\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$

$\vdots$

$A^4b=[b, Ab, A^2b, A^3b]\begin{bmatrix}-8\\20\\-18\\7\end{bmatrix}$

Thus, the representation of A w.r.t. $\{b, Ab, A^2b, A^3b\}$ is $\begin{bmatrix}0&0&0&-8\\1&0&0&20\\0&1&0&-18\\0&0&1&7\end{bmatrix}$; This happens to be the representation w.r.t. $\{\bar{b}, A\bar{b}, A^2\bar{b}, A^3\bar{b}\}$ as well.

3.14 $\det(\lambda I - A) = (\lambda + a_1)\cdot \det\begin{pmatrix} \lambda & 0 & 0 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{pmatrix} + 1\cdot \det\begin{pmatrix} a_2 & a_3 & a_4 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{pmatrix}$

$$= \lambda^3(\lambda + a_1) + a_2\lambda^2 + a_3\lambda + a_4$$

which is the stated characteristic polynomial.

Now, if $\det(\lambda_i I - A) = 0 \Rightarrow$

$$\lambda_i^4 = -a_1\lambda_i^3 - a_2\lambda_i^2 - a_3\lambda_i - a_4$$

Substituting in

$$A\begin{bmatrix} \lambda_i^3 \\ \lambda_i^2 \\ \lambda_i \\ 1 \end{bmatrix} \text{ we find } \lambda_i \begin{bmatrix} \lambda_i^3 \\ \lambda_i^2 \\ \lambda_i \\ 1 \end{bmatrix}$$

so $[\lambda_i^3, \lambda_i^2, \lambda_i, 1]$ is an eigenvector of A associated with λ_i.

3.15 $\det\begin{bmatrix} \lambda_1^3 & \cdots & \lambda_4^3 \\ \lambda_1^2 & \cdots & \lambda_4^2 \\ \lambda_1 & & \lambda_4 \\ 1 & & 1 \end{bmatrix} = \det\begin{bmatrix} \lambda_1^3 - \lambda_4^3 & & \lambda_4^3 \\ \lambda_1^2 - \lambda_4^2 & \cdots & \lambda_4^2 \\ \lambda_1 - \lambda_4 & & \lambda_4 \\ 0 & & 1 \end{bmatrix}$

$$= \det\begin{pmatrix} \lambda_1^3 - \lambda_4^3 & \cdots & \lambda_3^3 - \lambda_4^3 \\ \vdots & & \\ \lambda_1 - \lambda_4 & \cdots & \lambda_3 - \lambda_4 \end{pmatrix} =$$

$$= (\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4) \det\begin{bmatrix} \lambda_1^2 + \lambda_1\lambda_4 + \lambda_4^2 & \cdots \\ \lambda_1 + \lambda_4 & \\ 1 & \end{bmatrix}$$

common factors of column

$$= \prod_{1 \le j \le 4} (\lambda_j - \lambda_4) \left| \left[\text{Col 1} - \text{Col 2} \;\middle|\; \text{Col 2} - \text{Col 3} \;\middle|\; \begin{matrix} \lambda_3^2 + \lambda_3\lambda_4 + \lambda_4^2 \\ \lambda_3 + \lambda_4 \\ 1 \end{matrix} \right] \right|$$

Col 1 − Col 2 ↓

$$\begin{matrix} (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 + \lambda_4) & \cdots \\ \lambda_1 - \lambda_2 & \\ 0 & \end{matrix}$$

$$= (\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3) \begin{vmatrix} \lambda_1+\lambda_2+\lambda_4 & \lambda_2+\lambda_3+\lambda_4 \\ 1 & 1 \end{vmatrix}$$

$$= \prod_{1\leq i<j\leq 4} (\lambda_i-\lambda_j)$$

If all eigenvalues are distinct then the determinant is nonzero $\Rightarrow$ the columns are linearly independent.

<u>3.21</u> Using Cayley-Hamilton, functions of A that can be written as a power series, can be expressed in terms of $I, A, A^2 \ (= A^{n-1})$.

1. $\beta_0 + \beta_1 A + \beta_2 A^2 = A^{103} = f(A)$

Evaluating at the eigenvalues and their derivatives

$\lambda_1 = 0 \ : \quad f(0) = \beta_0 = 0^{103} = 0$

$\lambda_2 = 1 \ : \quad f(1) = \beta_0 + \beta_1 + \beta_2 = 1^{103} \Rightarrow \beta_1 + \beta_2 = 1$

$$\left.\frac{\partial f(\lambda_2)}{\partial \lambda_2}\right|_1 = \beta_1 + 2\beta_2 = 103 \cdot 1^{102} = 103$$

$$\Rightarrow A^{103} = -101A + 102A^2 = \begin{bmatrix} 1 & 1 & 102 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Similarly, $A^{10} = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

3. $e^{At} = \begin{bmatrix} e^t & e^t-1 & te^t - e^t + 1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{bmatrix}$

3.31 $AM + MB = C$

$A = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$ $B = 3$ $C = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ $\Rightarrow M = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$

$\text{eig}(A) = -1+j, -1-j$
$\text{eig}(B) = 3$
$\} \rightarrow \lambda_A + \lambda_B \neq 0$

∴ The Lyapunov equation is nonsingular and has a unique solution for any C.

Solving the Linearsystem, $m_1 = 0$, $m_2 = 3$.

3.36 The expression follow directly from the general case 3.69 (see reference for details)

For a more elementary proof, consider the LU decomposition of the matrix into a lower triangular and an upper triangular:

$$\det(A + UV^T) = \det(1 + V^T A^{-1} U) \det A$$

For $A = I$ verify that

$$\begin{pmatrix} 1 & 0 \\ V^T & 1 \end{pmatrix} \begin{pmatrix} 1 + UV^T & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -V^T & 1 \end{pmatrix} = \begin{pmatrix} 1 & U \\ 0 & 1 + V^T U \end{pmatrix}$$

Taking determinants

$$(1) \cdot (1 + UV^T)(1) = 1 + V^T U$$

This property is referred to as the Matrix Determinant Lemma (Sylvester) and it is related to the Matrix Inversion Lemma, which is useful in simplifying recursive updates in Least Squares estimation

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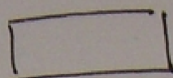
3.38 $Ax = y$, $A: m \times n$, rank $A = m \Rightarrow n \geq m$

$A:$ 

Then A^TA is an $n \times n$ matrix with rank m, so, its inverse is not defined and the given expression is not a solution.

It will be a solution if $n = m$, in which case A is invertible and $(A^TA)^{-1}A^Ty = A^{-1}y$

[Notice that if $m > n$ and rank$(A) = n$, $(A^TA)^{-1}A^Ty$ is still not a solution unless y is in the Range of A (e.g. rk(A) = rk$([A|y])$)]

On the other hand, $A^T(AA^T)^{-1}y$ is a solution and, in fact, it is the minimum norm solution. When $m = n$, this expression also reduces to $A^{-1}y$.

EEE 582 HW 3 SOLUTIONS

4.2 Laplace method: $y(s) = \frac{5s}{s^2+2s+2} u(s)$

$= \frac{5}{(s+1)^2+1}$

Using tables $y(t) = 5e^{-t} \sin t \quad (t \geq 0)$

STM method (matrix exponential)

$\text{eig}(A) = -1 \pm j$

$f(\lambda) = e^{\lambda t}$, $h(\lambda) = \beta_0 + \beta_1 \lambda$, evaluate at $\lambda = -1-j, -1+j$

$\Rightarrow \beta_0 = e^{-t} \sin t$, $\beta_1 = e^{-t}(\sin t + \cos t)$

$\Rightarrow e^{At} = \beta_0 I + \beta_1 A$

with $u(t) = 1$, $t \geq 0$ $\Rightarrow y(t) = [2, 3] \int e^{A(t-\tau)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot 1 \cdot d\tau$

$$y(t) = \int_0^t 5e^{-(t-\tau)} \cos(t-\tau) - 5e^{-(t-\tau)} \sin(t-\tau) \, d\tau$$

$= \ldots = 5e^{-t} \sin t.$

4.4 Companion form $Q = [b, Ab, A^2 b] = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 2 & -4 \\ 1 & -2 & 0 \end{bmatrix}$

$\bar{A} = Q^{-1} A Q = \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & -6 \\ 0 & 1 & -4 \end{bmatrix}$; $\bar{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Nodal form: Evals: $-1+j$, $-1-j$, -2

Evecs: $v_1 = \begin{bmatrix} 0 \\ 0.58j \\ -0.58-0.58j \end{bmatrix}$, $v_2 = v_1^*$, $v_3 = \begin{bmatrix} 0.71 \\ 0 \\ -0.71 \end{bmatrix}$

$Q = [\text{Re}\, v_1, \text{Im}\, v_1, v_3]$; $Q^{-1} A Q = \bar{A} = \begin{bmatrix} -1 & 1 & \\ -1 & 2 & \\ & & -2 \end{bmatrix}$, $\bar{B} = \begin{bmatrix} -3.46 \\ 0 \\ 1.41 \end{bmatrix}$

4.8 Equivalence (Algebraic) means that there is a coordinate transformation relating the two representations, so $Q^{-1}A_1Q = A_2 \Rightarrow A_1, A_2$ have the same eigenvalues. This is not the case here, so the two are not equivalent.

But, it follows from the direct computation of the transfer function $C(sI-A)^{-1}B$ that both realizations have the transfer function $\frac{1}{(s-2)^2}$, so they are zero-state equivalent.

4.11 1. Pull out the direct feed-through

$$G(s) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} \frac{2}{s+1} & \frac{2s-3}{(s+1)(s+2)} \\ -\frac{3}{s+1} & \frac{-2}{s+2} \end{bmatrix}$$

& factorize the common denominator

$$= \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \frac{1}{s^2+3s+2} \begin{bmatrix} 2s+4 & 2s-3 \\ -3s-6 & -2s-2 \end{bmatrix}$$

Using (4.34),

$$\dot{x} = \begin{bmatrix} -3 & 0 & -2 & 0 \\ 0 & -3 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 2 & 4 & -3 \\ -3 & -2 & -6 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} u.$$

(A 4-dim. realization)

4.12 Performing the realization in columns

$$G_{:,1} = \begin{bmatrix}0\\1\end{bmatrix} + \frac{1}{s+1}\begin{bmatrix}2\\-3\end{bmatrix} \rightarrow [A,B,C,D] = \left[-1,\ 1,\ \begin{bmatrix}2\\-3\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right]$$

$$G_{:,2} = \begin{bmatrix}0\\1\end{bmatrix} + \frac{1}{s^2+3s+2}\begin{bmatrix}2s-3\\-2s-2\end{bmatrix} \rightarrow$$

$$[A,B,C,D] = \left[\begin{bmatrix}-3 & -2\\1 & 0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2 & -3\\-2 & -2\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right]$$

Combining the two

$$\dot{x} = \begin{bmatrix}-1 & 0 & 0\\0 & -3 & -2\\0 & 1 & 0\end{bmatrix} x + \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix} u$$

$$y = \begin{bmatrix}2 & 2 & -3\\-3 & -2 & -2\end{bmatrix} x + \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix} u$$

Which is a 3-dimensional realization, one dim less than 4-11. Notice that a "brute-force" realization of each term and then concatenation of all state vectors would yield a 5-dimensional realization.

4.17 $$\frac{\partial}{\partial t}\Phi(t_0,t) = X(t_0)\frac{\partial}{\partial t}X^{-1}(t) \quad ; \quad X: \text{Fundamental Matrix.}$$

$$= X(t_0)\left(-X^{-1}(t)A(t)\right) \quad ; \quad \dot{X}^{-1} = -X^{-1}\dot{X}X^{-1}$$

$$= -\Phi(t_0,t)A(t)$$

4.21

$$X(t) = e^{At} C e^{Bt} \quad ; \quad X(0) = ICI = C$$

$$\dot{X} = Ae^{At} C e^{Bt} + e^{At} C e^{Bt} B$$

$$= AX(t) + X(t)B$$

By uniqueness of solutions of ODEs, X is the solution of

$$\dot{X} = AX + XB, \quad X(0) = C$$

4.24

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Consider the coordinate transformation $\bar{x} = P(t)x = e^{-At}x$

Then $\bar{A} = [P(t)A + \dot{P}(t)]P^{-1}(t)$

$$= (e^{-At}A - e^{-At}A)e^{At} = 0$$

$$\bar{B} = P(t)B = e^{-At}B$$

$$\bar{C} = CP^{-1}(t) = Ce^{At}$$

$\therefore (A, B, C) \sim (0, \bar{B}(t), \bar{C}(t))$ (Algebraic Equivalence)

EEE 582 HW # 4 SOLUTIONS

5.4 1) Impulse response $g(t) = \mathcal{L}^{-1}\{\hat{g}(s)\}$
$$= e^{-(t-2)} \quad (t \geq 2)$$

$\int |g(t)| dt = 1 < \infty \Rightarrow$ BIBO stable

2) $\hat{g}(s)$ is a cascade interconnection of two stable systems e^{-2s} (shift/delay) and $\frac{1}{s+1}$
$\therefore$ stable

5.7
$$\hat{g}(s) = [-2, 3] \begin{bmatrix} s+1 & -10 \\ 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \frac{4}{s+1}$$

$\Rightarrow$ BIBO stable.

(Notice that the unstable mode $\frac{1}{s-1}$ is canceled)

5.10 $\sigma(A) = \{-1, 0, 0\} \Rightarrow$ not Asymptotically stable.

For marginal stability, we should have the max Jordan block for $\{0\}$-eigenvalue to have size 1.

$$\text{Null}(0I - A) = \text{Null}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \dim 2$$

$\Rightarrow$ there are 2 Jordan blocks for 0 so their size is 1 $\Rightarrow$ the homogeneous eqn is marginally stable

---

5.11 Just as in 5.10 $\dim \text{Null}(0I - A) = \dim \text{Null}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = 1$

$\Rightarrow$ there is 1 Jordan block for 0 $\Rightarrow$ the homogeneous eqn. is not stable.

5.12 This is the discrete time analog of 5.10.
There are three eigenvalues, 0.9, 1, 1 ; the first is A.S.
The two at 1 are marginally stable iff they correspond to different Jordan blocks. Since dim Null $(1I-A) = 2$, there are 2 Jordan blocks for $\{1\}$ $\Rightarrow$ equ. is marginally stable.

5.14 $A^T M + MA = -I \Rightarrow M = \begin{bmatrix} 1.75 & 1 \\ 1 & 1.5 \end{bmatrix}$

Hurwitz test: $1.75 > 0$

$\det\begin{pmatrix} 1.75 & 1 \\ 1 & 1.5 \end{pmatrix} = 1.75 \times 1.5 - 1 > 0$

$\Rightarrow M > 0 \Rightarrow \operatorname{Re} \operatorname{eig}(A) < 0.$

5.18 $A^T M + MA + 2\mu M = -N \Rightarrow (A+\mu I)^T M + M(A+\mu I) = -N$

since $M, N > 0$, we have $\operatorname{Re} \operatorname{eig}(A+\mu I) < 0$

$\Rightarrow \operatorname{Re} \operatorname{eig}(A) < -\mu$

5.19 $\rho^2 M - A^T M A = \rho^2 N \Rightarrow M - (\frac{1}{\rho}A)^T M (\frac{1}{\rho}A) = N$

$\Rightarrow |\operatorname{eig}(A \cdot 1/\rho)| < 1 \Rightarrow |\operatorname{eig}(A)| < \rho.$

EEE 582 HW 5 SOLUTIONS

6.2 $[B, AB] = \begin{bmatrix} 0 & 1 & \vdots & 1 & 0 \\ 1 & 0 & \vdots & 0 & 0 \\ 0 & 0 & \vdots & 2 & 0 \end{bmatrix}$ has full row rank

$\Rightarrow$ completely controllable (A^2B not needed here)

$\begin{bmatrix} C \\ AC \\ A^2C \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & -2 & 4 \end{bmatrix}$ has rank 3

$\Rightarrow$ completely observable

6.4 (A, B) is c.c. $\Leftrightarrow$ $\text{rank}\begin{bmatrix} A_{11}-sI & A_{12} & B_1 \\ A_{21} & A_{22}-sI & 0 \end{bmatrix} = n$

for all s. Hence $[A_{21}, A_{22}-sI]$ should have full row rank for all s $\Leftrightarrow$ (A_{22}, A_{21}) is c.c.

6.8 $\dot{x} = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad y = [1 \ 1] x$

$Q_C = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$. Select $P^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$\Rightarrow P^{-1}AP = \begin{bmatrix} 3 & 4 \\ 0 & -5 \end{bmatrix} \quad PB = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad CP^{-1} = [2, 1]$

$\Rightarrow \dot{\bar{x}} = \begin{bmatrix} 3 & 4 \\ 0 & -5 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad y = [2, 1]\bar{x}$

Obviously, the equation can be reduced (zero state equivalent system) to: $\dot{x}_1 = 3x_1 + 4, \quad y = 2x_1$ which is c.o.

6.15 For controllability (JCF test) the rows $\begin{bmatrix} b_{21} & b_{22} \\ b_{41} & b_{42} \\ b_{61} & b_{52} \end{bmatrix}$ should be independent; this is not possible.

For observability, the columns of $\begin{bmatrix} c_{11} & c_{13} & c_{15} \\ c_{21} & c_{23} & c_{25} \\ c_{31} & c_{33} & c_{35} \end{bmatrix}$ should be independent; this is possible e.g. I (identity)

(7.56)

7.10 $a_1 = 2, a_2 = 1, h(1) = 0, h(2) = 1 \Rightarrow$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

This is a companion form realization.

7.11 $T = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$ $\underset{(TT)}{\tilde{T}} = \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}$

Using MATLAB,

```
[k,s,L] = svd(T)
 s1 = sqrtm(s)
QO = k * S1 ; QC = S1 * L ;
a = inv(QO) * TT * inv(QC);
b = QC(1:2,1) ; c = QO(1,1:2)
```

$$\Rightarrow \dot{x} = \begin{bmatrix} -1.707 & 0.707 \\ -0.707 & -0.293 \end{bmatrix} x + \begin{bmatrix} 0.595 \\ -0.595 \end{bmatrix} u$$

$$y = \begin{bmatrix} -0.595, & -0.595 \end{bmatrix} x$$

This is a balanced realization in the sense of the controllability/observability matrices.

EEE 582 HW #6 SOLUTIONS

#8.1

$$\dot{x} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u \qquad y = [1\ 1]\, x$$

$$u = r - [k_1\ k_2]\, x$$

$$\Rightarrow \dot{x} = \underbrace{\begin{bmatrix} 2-k_1 & 1-k_2 \\ -1-2k_1 & 1-2k_2 \end{bmatrix}}_{A_c} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} r$$

$$\det(sI - A_c) = s^2 + ((k_1+2k_2)-3)s + (k_1 - 5k_2 + 3) \quad \begin{cases} k_1 + 2k_2 - 3 = 3 \\ k_1 - 5k_2 + 3 = 2 \end{cases}$$

$$= \Delta_f(s) = (s+1)(s+2) = s^2 + 3s + 2 \Rightarrow$$

$$\Rightarrow \boxed{k_1 = 4,\ k_2 = 1}$$

#8.2

$$\Delta(s) = \det(sI - A) = s^2 - 3s + 3 \Rightarrow \bar{k} = [3+3,\ 2-3] = [6, -1]$$

(in cc. form coordinates)

$$\bar{Q}_c^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \quad \bar{Q}_c = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$ controllability matrix in cc form coordinates

$$Q_c = [b\ \ Ab] = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}, \quad Q_c^{-1} = -\frac{1}{7}\begin{bmatrix} 1 & -4 \\ -2 & 1 \end{bmatrix}$$

$$k = \bar{k}\,\bar{Q}_c Q_c^{-1} = [6\ -1]\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}(-1/7)\begin{bmatrix} 1 & -4 \\ -2 & 1 \end{bmatrix} = \boxed{[4, 1]}$$

#8.7

$$\dot{x} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u, \quad y = [2, 0, 0]\, x$$

$$\left.\begin{aligned} \det(sI - A) &= s^3 - 3s^2 + 3s - 1 \\ \Delta_f(s) &= (s+2)(s+1+j)(s+1-j) = s^3 + 4s^2 + 6s + 4 \end{aligned}\right\} \Rightarrow \bar{k} = [7, 3, 5]$$

$$\bar{Q}_c = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad Q_c^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -3 & 2 \\ 1 & 2 & -1 \end{bmatrix} \Rightarrow k = \bar{k}\,\bar{Q}_c Q_c^{-1} = \boxed{[15, 47, -8]}$$

To track steps in r, we define $u = pr - kx$ and compute p s.t. the transfer function $r \to y$ has unity DC gain:

$$y = c(sI - A + Bk)^{-1} B r \ ; \ G_c(0) = [2\,0\,0]\left[-\begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}[15, 47, -8]\right]^{-1}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} p = 1$$

$$\Rightarrow \boxed{p = 0.5}$$

8.8 $x_{k+1} = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} u_k, \quad y_k = [2\ 0\ 0]\, x_k$

$\det(zI - A) = (z-1)^3 = z^3 - 3z^2 + 3z - 1$
$\Delta_f(z) = z^3 = z^3 + 0z^2 + 0z + 0$
$\Rightarrow \bar{k} = [3, -3, 1]$

$Q_c, \bar{Q}_c$ are the same as in Pr#8.7 $\Rightarrow k = \bar{k}\bar{Q}_c Q_c^{-1} = [1, 5, 2]$

The closed-loop system is

$$x_{k+1} = (A - Bk)\, x_k + B u_k, \quad y_k = C x_k$$

where $(A - Bk) = \begin{pmatrix} 0 & -4 & -4 \\ 0 & 1 & 1 \\ -1 & -5 & -1 \end{pmatrix}$

The zero input response is $x_k = (A - Bk)^k x_0$

we compute $(A - Bk)^2 = \begin{pmatrix} 4 & 16 & 0 \\ -1 & -4 & 0 \\ 1 & 4 & 0 \end{pmatrix}$, $(A - Bk)^3 = 0$

$\therefore$ for any x_0, $x_k = 0$ for $k \geq 3$

Note: since $(A - Bk, B)$ is controllable with 1 input and
$\text{eig}(A - Bk) = 0, 0, 0 \Rightarrow \text{JCF}(A - Bk) = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$, which is nilpotent
with $k = 3$ $(A - Bk)^k = (E^{-1}\, \text{JCF}(A - Bk)\, E)^k = E^{-k}\, [\text{JCF}(A - Bk)]^k E^k = 0$
for $k \geq 3$

HW # Minimal Realization, Solutions

1. The minimal realization has dimension 4.

2. The Gramians of the balanced realization are diagonals with entries

9.5135e-001 1.5584e-001 8.9122e-002 1.4599e-002

We can, therefore, eliminate the last balanced state with additive error at most

2*1.4599e-002 = 0.029

and the last two states with additive error at most

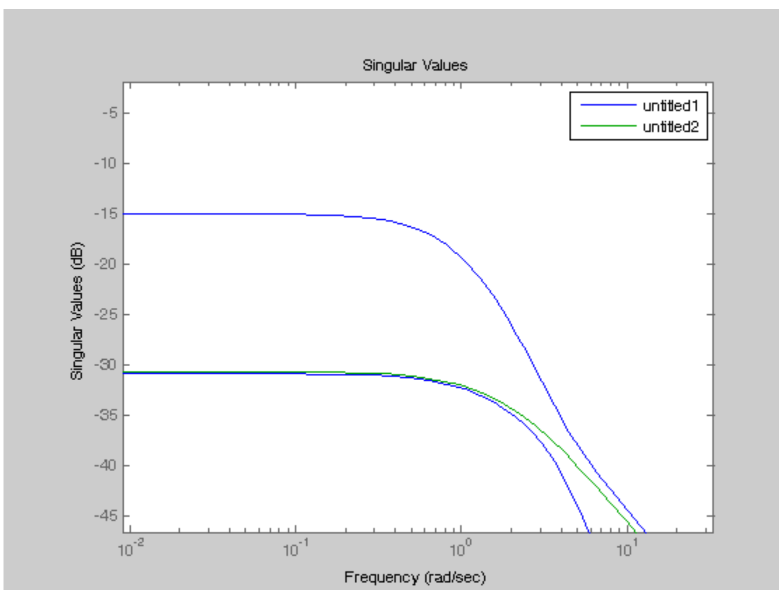
2*(8.9122e-002 + 1.4599e-002) = 0.207

Let H be the original system, Hm the minimal realization, Hb the balanced realization, Hb2 the balanced truncation with two states and Hb3 the balanced truncation with three states. The maximum singular value of the difference transfer function is the induced L-2 norm of the difference (error) system. We can plot the singular values in MATLAB using sigma.

Hb2=ss(a(1:2,1:2),b(1:2,:),c(:,1:2), d), Hb3=ss(a(1:3,1:3),b(1:3,:),c(:,1:3),d)

sigma(H-Hm,Hm-Hb,Hb-Hb2,Hb-Hb3)

The first two are numerically the same so the singular values of the difference are numerically zero. Hb and Hb3 are different only in one state so only one singular value is essentially different from zero. Zooming in, we observe that the peak sigma (Hb-Hb2) is -15dB ~ 0.178 (<=0.207) and the peak sigma (Hb-Hb3) is -30.5dB ~ 0.029 (<=0.029), as expected.



EEE 582 **HW # SVD**

```
% SVD application in the modeling of a noisy oscillatory signal
%     as the output of an autoregressive model:
%     y(n)=[y(n-1),y(n-2),...]*q

% Define the signals
t=(0:.01:10)';           % time
n=rand(size(t))-.5;      % noise
y=sin(2*t+1)+sin(10*t+1)+.02*n/2;    % measurement
N0=10:length(t);         % fitting window
F=tf(.01,[1 -.99],.01);y=lsim(F,y); % Optional Filtering for frequency-weighted fit

% Form the regressor by taking lags of the output
W=[y(N0-1),y(N0-2),y(N0-3),y(N0-4),y(N0-5),y(N0-6),y(N0-7),y(N0-8),y(N0-9)];
NN=0.02*[n(N0-1),n(N0-2),n(N0-3),n(N0-4),n(N0-5),n(N0-6),n(N0-7),n(N0-8),n(N0-9)];
q=W\(y(N0)), % least squares fit
plot(t,y,t(N0),W*q,t(N0),W*q-y(N0)); pause    % check the fit

% Autoregressive transfer function: resonance at the oscillation frequency
g=tf(1,[1 -q'],.01), bode(g)                  % check the t.f.
% Model order: How many columns of W do you need?
s=svd(W)            % svd of regressor
s2=svd(W'*W)         % svd of gramian

% Questions:
% 1. What is the relationship between s and s2? How many lags do you need
% in the model?
% 2. The singular values of W appear to reach a floor related to the noise.
%    Derive this value analytically and verify with an example.
% 3. What is ithe effect of the noise amplitude?
% 4. What happens when the signal is composed of two frequencies, say 10
% and 2?
```

Answers

1. $s = \sqrt{s2}$; we need at least two lags (2nd order difference equation) to describe a sinusoidal solution.

2. Letting W_0 be the deterministic component and n be the noise, $W = W_0+n$, $W^TW = (W_0+n)^T(W_0+n) = W_0^TW_0 + n^Tn$ since $n \perp W$ (noise uncorrelated with signal). Furthermore, since each sample of the noise is independent, $n^Tn = N\rho I$, where ρ is the noise variance and N is the number of points. For the uniform distribution in the interval $[a,b]$, with mean $\bar{n} = (a+b)/2$, $var(n) = \frac{1}{b-a}\int_a^b (n-\bar{n})^2 dn = (b-a)^2/12$. In our case the distribution is symmetric, zero mean. Let r denote the maximum amplitude. Then, $\rho = r^2/3$. In the program, $r = 0.01$, $N = 1001$, so $rho = 0.033$. The small eigenvalues of $s2$ range in 0.0355 to 0.0316, which agrees with the theoretical value.

3. Increasing the noise amplitude increases the singular values of W and introduces a bias. Denoting by '$\sharp$' the left inverse (LS solution), we have

$$q = W^\sharp y = (W_0+n)^\sharp (y_0 + n_{k+1}) = (W_0^TW_0 + \rho I)^{-1} W_0^T y_0$$

The higher the ρ, the more the solution deviates from the nominal one $(W_0^TW_0)^{-1}W_0^Ty_0$ For example, we find by trial and error that when the noise amplitude is 0.45, the resonance is smeared and is bearly recognizable.

4. When two frequencies are present, the identification is more difficult. With noise amplitude 0.01, the frequency 2 is not recognized (because it does not possess enough energy-cycles in the data interval). Reducing the noise amplitude by more than a factor of 50 allows for the second peak to appear in the fitted model. Alternatively, introducing a low-pass filter to pre-process the data has a similar effect since it attenuates the noise at high frequencies and effectively reduces the variance entering the regressor matrix.