EEE 587 Test 1 SOLUTIONS

① $\min_x |x|^2$ s.t. $Ax=b$ $\Leftrightarrow$ $\min_{x,\lambda} x^Tx + \lambda(Ax-b) \overset{\text{NEC}}{\Rightarrow} \begin{cases} \partial/\partial x & 2x^T + \lambda^T A = 0 \quad ① \\ \partial/\partial \lambda & Ax = b \quad ② \end{cases}$

$\underset{\text{sub. ① into ②}}{\Rightarrow}$ $2Ax + AA^T\lambda = 0,\ Ax=b \Rightarrow \lambda = (AA^T)^{-1}(-2b)$ ← invertible

$\overset{②}{\Rightarrow} x = A^T(AA^T)^{-1}b$, Assuming that AA^T is invertible

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② $\mathcal{H} = \frac{1}{2}(x^Tx + u^Tu) + \lambda^T(Ax+Bu)$

$\min_u \mathcal{H}$: $\frac{\partial \mathcal{H}}{\partial u} = 0 \Rightarrow u = -B^T\lambda$

$-\dot{\lambda} = \frac{\partial \mathcal{H}}{\partial x} = x + A^T\lambda$

$\dot{x} = Ax + Bu$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \underbrace{\begin{pmatrix} A & -BB^T \\ -I & -A^T \end{pmatrix}}_{H} \begin{pmatrix} x \\ \lambda \end{pmatrix}$$

with BC (transversality) $\left(\frac{\partial S}{\partial t} + \mathcal{H}\right)\Big|_{t_f} \delta t_f + \left(\frac{\partial S}{\partial x} + (-\lambda^T)\right)\Big|_{t_f} \delta x_f = 0$

$\therefore \begin{pmatrix} x \\ \lambda \end{pmatrix}(t) = \Phi \begin{pmatrix} x \\ \lambda \end{pmatrix}(0)$ where $\Phi = e^{H(t_f - t_0)}$

1. BC: $x(0) = 0,\ x(1) = [1;1] \Rightarrow \begin{pmatrix} 1 \\ 1 \\ \lambda_{1f} \\ \lambda_{2f} \end{pmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ \lambda_{10} \\ \lambda_{20} \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \Phi_{12} \begin{pmatrix} \lambda_{10} \\ \lambda_{20} \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda_{10} \\ \lambda_{20} \end{pmatrix} = \Phi_{12}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

2. BC: $x(0) = 0 \quad x(1) = [1, x_2(1)]$ $\quad \frac{\partial S}{\partial t} = 0,\ \frac{\partial S}{\partial x}^T = F x(1)$

$\Rightarrow (\overset{I}{\cancel{F}} x(1) - \lambda)^T \begin{pmatrix} \cancel{\delta x_{f1}} \\ \delta x_{f2} \end{pmatrix} = 0 \Rightarrow x_2(1) - \lambda_2(1) = 0$

$\Rightarrow \begin{pmatrix} 1 \\ x_{2f} \\ \lambda_{1f} \\ \lambda_{2f} \end{pmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ \lambda_{10} \\ \lambda_{20} \end{pmatrix}$

$0 = x_{2f} - \lambda_{2f}$

$\Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} [1, 0]\,\Phi_{12} \\ [0, 1](\Phi_{12} - \Phi_{22}) \end{bmatrix} \begin{pmatrix} \lambda_{10} \\ \lambda_{20} \end{pmatrix}$

$\Rightarrow \begin{pmatrix} \lambda_{10} \\ \lambda_{20} \end{pmatrix} = \ldots$

③ Hamiltonian $\mathcal{H} = u^2 + \lambda^T(ax+u)$

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \Rightarrow u = -\tfrac{1}{2}\lambda$$

$$\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{x} = ax + u$$

$$\frac{\partial \mathcal{H}}{\partial x} = -\dot{\lambda} = a\lambda \quad \Rightarrow \lambda(t) = e^{-at}\lambda(0)$$

$$\Rightarrow u(t) = -e^{-at}\lambda_0/2$$

$$\Rightarrow \underset{0}{x(1)} = e^{a}x_0 + \int_0^1 e^{a}e^{-a\tau}\left(-e^{-a\tau}\lambda_0/2\right)d\tau$$

$$\Rightarrow \lambda_0 = \frac{4a}{1-e^{-2a}}$$

Opt. Energy $= \int_0^1 u^2\,dt = \frac{\lambda_0^2}{4}\cdot\int_0^1 e^{-2at}\,dt = \boxed{\frac{e^{-2a}-1}{2a}}$

Constant Control: u_c : $-e^{a1}\cdot 1 = \int_0^1 e^{a1}e^{-a\tau}u_c\,d\tau \Rightarrow u_c = -\frac{a}{1-e^{-a}}$

Energy: $\int_0^1 u_c^2 = \boxed{\frac{a^2}{(1-e^{-a})^2}}$

$\therefore \frac{En_{,const}}{En_{,opt}} = \frac{a}{2}\frac{(1-e^{-2a})}{(1-e^{-a})^2} = R$