EEE 587 HW 4 SOLUTIONS

Pr 7.3

$$\dot{x} = Ax + Bu \; ; \quad \min J = \int_0^{t_f} 1 \, dt$$

$$x_0 \rightarrow [2,2] = x_f$$

$$H = 1 + \lambda^T (Ax + Bu)$$

$\Rightarrow$ optimal control: $u = -\text{sign}\, B^T\lambda$

$\dot{\lambda} = -A^T\lambda$. (integrators $\Rightarrow$ max 1 switching)

Optimal policy: find the switching surface. Then

$u_*(x) = +u_{Max}$ if x below the surface

$u_*(x) = -u_{Max}$ if x above " " .

The switching surface becomes more "vertical" when u_{max} increases.

Computational detail: to determine the switching surface solve $\dot{x} = Ax + Bu$ backwards in time, starting with I.C. [2;2]. I.e.,

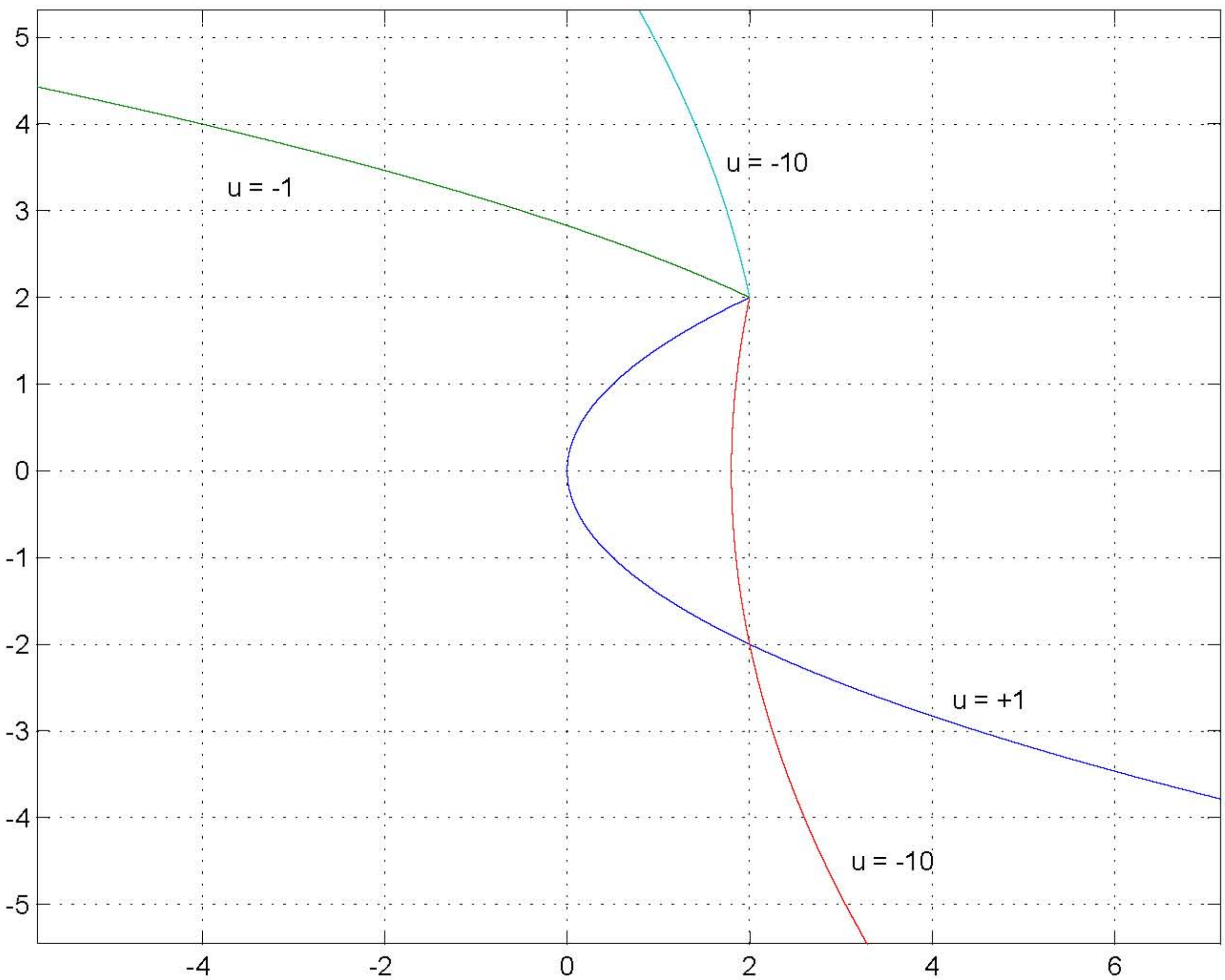
x = lsim(-A, -B, I, 0, ones(size(t)), t, [2;2])

When the control is unconstrained $\min t_f = 0$ (can infimum, actually!) and the optimal control becomes impulsive.

This can also be seen from $\min J(t_f) = \frac{1}{2}\int_0^{t_f} u^T u$.

The optimal control exists for arbitrarily small t_f, and can be computed from
$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & -BB^T \\ 0 & -A^T \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix}$$
with B.C. x_0, x_f ($\Rightarrow \lambda_0 = \Phi_{12}^{-1}(x_f - \Phi_{11} x_0) \ldots$)

Pr. 7.3, Switching surface for different input amplitudes

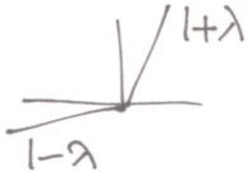


Pr. 7.6 $\quad \dot{x} = u, \quad |u| \le 1 \quad J = \int_0^{t_f} |u|, \ t_f$ free.

$\min J, \quad x_0 \to 0.$

$$H = |u| + \lambda u \quad \Rightarrow \quad u_{opt} = \begin{cases} -1 & \text{if } \lambda > 1 \\ 0 & \text{if } \lambda \in [-1, 1] \\ 1 & \text{if } \lambda < -1 \\ ? & \text{if } \lambda = \pm 1 \end{cases}$$

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = 0$$



t_f is free, system and objective are autonomous

$\Rightarrow H_{opt} = 0 \Rightarrow \lambda = \pm 1$ (singular interval)

Notice: $A = 0 \Rightarrow$ singular intervals may exist.

Indeed, the solution is not unique. For any t_f,

$u = -\dfrac{x_0}{t_f}$ satisfies the B.C. and $J = \int_0^{t_f} \dfrac{|x_0|}{t_f} \, dt = |x_0|$

(all have the same, optimal cost).

Pr. 7.8 $\quad \dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -a \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \quad ; \quad a > 0$, $|u| \le 1$

$$J = \int_0^{t_f} \beta + |u| \quad ; \quad t_f \text{ free}, \quad \beta > 0$$

$x_0 \to 0$

(weighted min-time - min fuel problem)

$$H = \beta + |u| + \lambda^T (Ax + Bu)$$

$$u_{opt} = \begin{cases} -1 & \text{if } \lambda_2 > 1 \\ 0 & \text{if } \lambda_2 \in [-1, 1] \\ 1 & \text{if } \lambda_2 < -1 \\ ? & \text{if } \lambda_2 = \pm 1 \end{cases} \qquad -\dot{\lambda} = \frac{\partial H}{\partial x} = A^T \lambda$$

$$\Rightarrow \dot{\lambda}_1 = 0 \qquad \dot{\lambda}_2 = -\lambda_1 + a\lambda_2 \Rightarrow \lambda(t) = \begin{bmatrix} \lambda_{10} \\ e^{at}\lambda_{20} + \frac{\lambda_{10}}{a}[1 - e^{at}] \end{bmatrix}$$

Depending on the sign and magnitude of the initial costates we may have up to two switchings ($-1 \to 0 \to 1$ or vice-versa). There can be no singular Intervals.

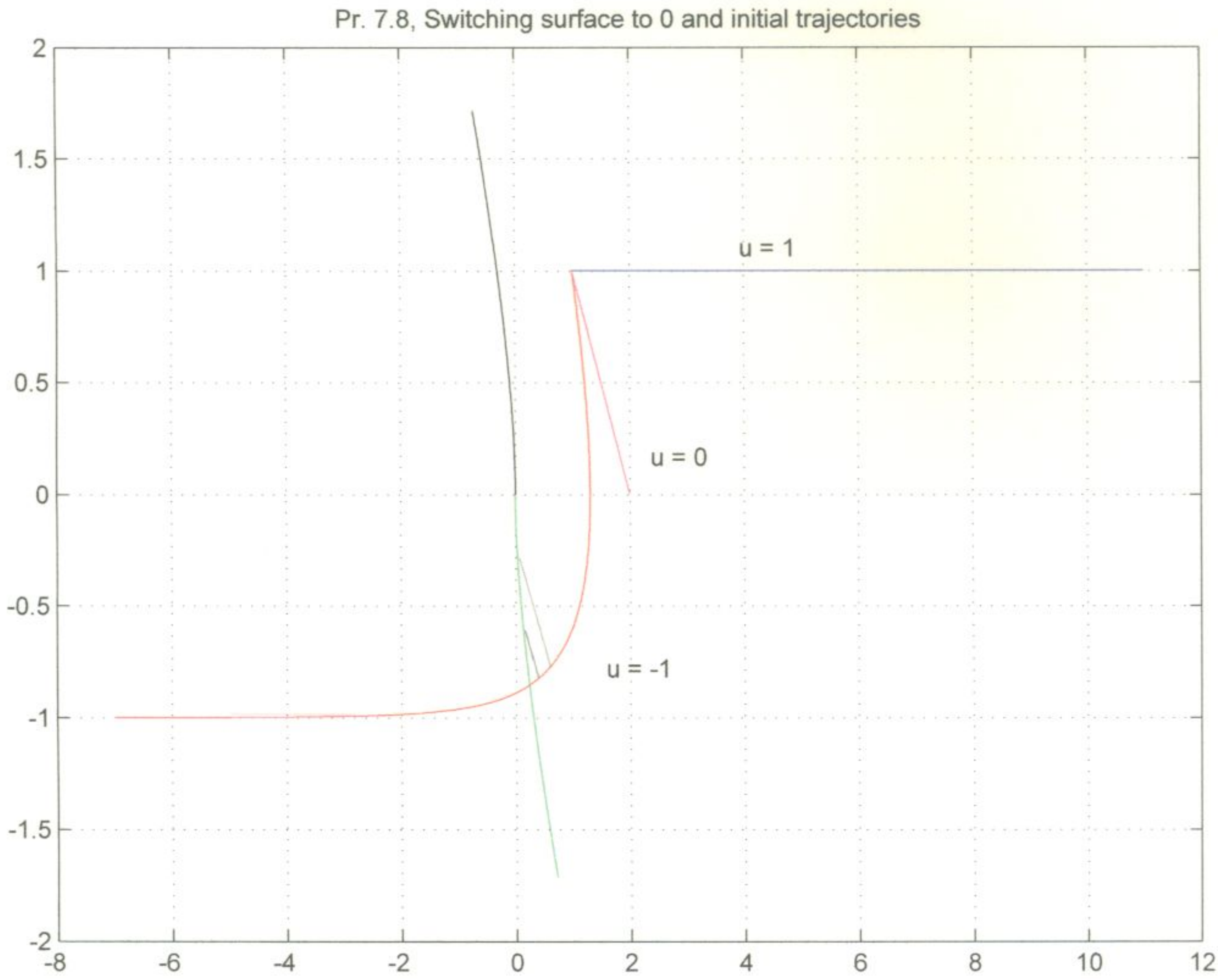
Critical values for λ, x at the switching times can be obtained from $H = 0$ e.g. switching

$$-1 \to 0 : \quad \lambda_2 = 1 \ @ \ t^- \quad u = -1 \Rightarrow \beta + 1 + (\lambda_{10}, 1)\left(Ax + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = 0$$

$$@ \ t^+ \quad u = 0 \Rightarrow \beta + (\lambda_{10}, 1) Ax = 0$$

etc.

Alternatively, we can sketch the switching surface to 0 with inputs ± 1, and the 3 possible trajectories from x_0 (ie. $u = +1, -1, 0$). Note: for $u = 0$, $x_1(t) = c - \frac{x_2(t)}{a}$, a straight line.



The time-optimal trajectory ($\beta \to \infty$) switches $\pm 1 \to \mp 1$ (red-green in our plot). The weighted time-fuel optimal connects the two segments with a straight line of slope $-a$. The length of the segment depends on β (but not straightforward to compute)

Pr 7.9 $\quad \dot{x} = \begin{pmatrix}0&1\\0&0\end{pmatrix} x + \begin{pmatrix}0\\1\end{pmatrix} u \qquad |u| \le 1$

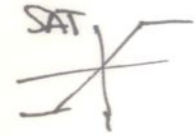
$$x_0 = (1;1) \longrightarrow x_f = (0;0) \ ; \ t_f = 4$$

$$J = \frac{1}{2}\int_0^4 u^2$$

$$H = \frac{1}{2}u^2 + \lambda^T(Ax + Bu)$$

u_{opt} : $\quad \frac{2u}{2} + B^T\lambda = 0 \Rightarrow u = -\frac{2}{2}B^T\lambda$ (uncoustr.)

when $|u| \le 1$: $\quad u_{opt} = \text{sat}\left[-\frac{2}{2}\underbrace{B^T\lambda}_{\lambda_2}\right]$;



$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -A^T x \Rightarrow \begin{array}{l}\lambda_1 = c_1 \\ \lambda_2 = c_2 - \lambda_1 t\end{array} \Rightarrow \lambda = \begin{pmatrix}1 & 0\\ -t & 1\end{pmatrix}\lambda_0$$

The uncoustrained solution is given by

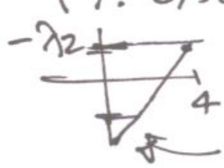
$u_{opt} = -\lambda_2(t)$ and

$$\begin{pmatrix}\dot{x}\\ \dot{\lambda}\end{pmatrix} = \underbrace{\begin{pmatrix}A & -BB^T\\ 0 & -A^T\end{pmatrix}}_{\hat{H}}\begin{pmatrix}x\\ \lambda\end{pmatrix} \Rightarrow \begin{pmatrix}x(t)\\ \lambda(t)\end{pmatrix} = \underbrace{e^{\hat{H}t}}_{\Phi}\begin{pmatrix}x_0\\ \lambda_0\end{pmatrix}$$

with $x_0 = \begin{pmatrix}1\\1\end{pmatrix}$, $x_f(4) = 0$

$$\Rightarrow \lambda_0 = -\Phi_{12}^{-1}(4)\,\Phi_{11}\,x_0 = \begin{pmatrix}0.5625\\ 1.3750\end{pmatrix}$$

$\lambda_2(t) = (t, -1)\,\lambda_0 \rightarrow$



violates the bound

$\Rightarrow$ the soln. is not uncoustrained $\Rightarrow$ it changes entirely. Without solving the TPBVP, we can observe that the soln will be equal to the uncoustrained ($u_{opt} = -\lambda_2$) after some time t_1 (if the problem is feasible).

Then, in $(0, t_1)$ $\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & \\ & -A^T \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} (-1)$

and in $(t_1, 4)$ $\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \hat{H} \begin{pmatrix} x \\ \lambda \end{pmatrix}$

with continuity at t_1 being the boundary condition.

In the interval $(0, t_1)$, $x(t) = \begin{pmatrix} x_{10} - \frac{1}{2}t^2 + x_{20}t \\ x_{20} - t \end{pmatrix}$

where $x_{10} = x_{20} = 1$.

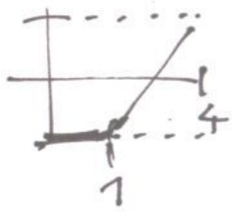
In the interval $(t_1, 4)$, $\begin{pmatrix} x \\ \lambda \end{pmatrix}(t) = e^{H(t-t_1)} \begin{pmatrix} x \\ \lambda \end{pmatrix}(t_1)$

$$\Rightarrow \lambda(t_1) = -\Phi_{12}^{-1}(4-t_1)\,\Phi_{11}(4-t_1)\,x(t_1)$$

$$\Rightarrow u(t) = -\lambda_2(t_1) = [(t-t_1), -1]\,\Phi_{12}^{-1}(4-t_1)\,\Phi_{11}(4-t_1) \cdot \begin{bmatrix} 1 - t_1^2/2 + t_1 \\ 1 - t_1 \end{bmatrix}$$

It is now a 1-parameter search to find t_1 s.t. $|u(t_1)| \leq 1$. The first (and only) time is at $t_1 = 1$. At that time $x(t_1) = \begin{pmatrix} 3/2 \\ 0 \end{pmatrix}$

we compute & sketch the rest of the solution to check if $|u| \leq 1$ $\forall t$.



The optimal cost is $J_* = 1$

(analytically or numerically)

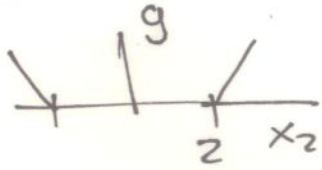
Pr 7.11

$$\dot{x} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u$$

$$J = \int 1$$

$$|u| \le 1, \quad |x_2| \le 2 \quad \text{(Velocity constraint)}$$

Let $g(x_2 - 2)$ be such that



Define the new state $\dot{z} = g^2$ and the isoperimetric constraint $z(0) = 0$, $z(t_f) = 0$.

Then, the Hamiltonian is

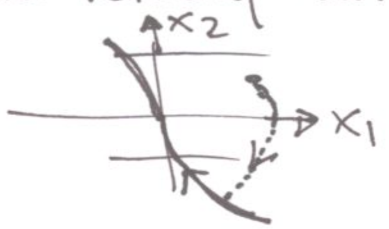
$$H = 1 + \lambda^T(Ax + Bu) + \mu g^2 \quad ; \quad \lambda, \mu : \text{costates}$$

Optimality: $\quad u_* = -\,\text{sign}\,(B^T\lambda)$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -A^T\lambda + 2\mu g \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\dot{\mu} = 0$$

When $g \equiv 0$, so that the solution satisfies the constraints, $\dot{\lambda} = -A^T\lambda \Rightarrow u_*$ is the time optimal input. This soln is Bang-Bang and may violate the velocity constraint. Intuitively, the optimal



soln. will use the ± 1 controls when $|x_2| < 2$ and 0 control (constant velocity) when $x_2 = 2$

The way this solution arises from our equations is by considering some time where $x_2 > 0$, but small. Then for large μ., x_2 will change sign very quickly. Thus, the solution will chatter around the constraint, which is expected since the extremals still have $u = \pm 1$. In an average sense, the trajectory moves in the correct direction (intuitively the same as $u=0$), see Fillipov for thry. Notice that any $x_2 > 0$ will make $z_2(t_f) > 0$.

But for $\mu \to \infty$, $z(t_f) \downarrow 0$.

PROBLEM 6.3

Build the table with entries:

STAGE, STATE & OPTIMAL COST-TO-GO ($J_*(x)$),

STATE & OPTIMAL CONTROL ($u_*(x)$).

Starting from the last stage we get:

STAGE	STATE	$J_*(x)$	u_*	NOTES
5	N	0	N-N	
4	H	3	H-N	
	I	4	I-N	
3	E	10	E-H	
	F	6	$\begin{Bmatrix} F-H \\ F-I \end{Bmatrix}$	$J_*(F) = \min\begin{Bmatrix} J_*(H)+3 \\ J_*(I)+2 \end{Bmatrix} = 6$ Multiple minima.
	G	9	G-I	
2	C	12	CF	$J_*(C) = \min\begin{Bmatrix} J_*(E)+4 \\ J_*(F)+6 \end{Bmatrix} = \min\begin{pmatrix} 14 \\ 12 \end{pmatrix}$
	D	13	D-F	$J_*(D) = \min\begin{pmatrix} J_*(G)+8 \\ J_*(F)+7 \end{pmatrix}$
1	L	15	$\begin{Bmatrix} LC \\ L-D \end{Bmatrix}$	$J_*(L) = \min\begin{Bmatrix} J_*(C)+3 \\ J_*(D)+2 \end{Bmatrix}$

Minimum Cost L - N = 15

Optimal paths:

$$\left|\begin{array}{l} L-C-F-H-N \\ L-C-F-I-N \\ L-D-F-H-N \\ L-D-F-I-N \end{array}\right\}$$ (just go through F)