**Pr 7.3**

\( \dot{x} = Ax + Bu \); \( \operatorname{min} \ J = \int_{0}^{t_f} 1 \, dt \)

\( x_0 \rightarrow [2,2] = x_f \)

\( H = 1 + x^T(Ax + Bu) \)

\( \Rightarrow \) **optimal control:** \( u = -\text{sign } B^T \lambda \)

\( \dot{x} = -A^T \lambda. (\text{integrators } \Rightarrow \text{max 1 switching}) \)

**Optimal policy:** find the switching surface. Then

\( u^+(x) = +u_{\text{Max}} \) if \( x \) below the surface

\( u^-(x) = -u_{\text{Max}} \) if \( x \) above 

The switching surface becomes more "vertical" when \( u_{\text{Max}} \) increases.

**Computational detail:** to determine the switching surface solve \( \dot{x} = Ax + Bu \) backwards in time, starting with I.C. \([2,2]\). I.e.,

\( x = l_{\text{sim}} \left( -A, -B, I, 0, \text{ones(size(t)), t, [2,2]} \right) \)

When the control is unconstrained \( \min_{t_f} = 0 \) (can infimum, actually!) and the optimal control becomes impulsive.

This can also be seen from \( \min \ J(t_f) = \frac{1}{2} \int_{0}^{t_f} u^T u \).

The optimal control exists for arbitrarily small \( t_f \), and can be computed from

\[
\begin{pmatrix}
\dot{x} \\
\lambda
\end{pmatrix} =
\begin{pmatrix}
A & -BB^T \\
0 & -A^T
\end{pmatrix}
\begin{pmatrix}
x \\
\lambda
\end{pmatrix}
\]

with B.C. \( x_0, x_f \) \( (\Rightarrow \phi_0 = \phi_{12}^{-1}(x_f - \phi_{11} x_0)) \)
Pr. 7.3, Switching surface for different input amplitudes

- $u = -1$
- $u = -10$
- $u = +1$
- $u = -10$
Pr. 7.6 $\dot{x} = u, \quad |u| \leq 1 \quad J = \int_{0}^{t_f} |u| \, dt, \quad t_f \text{ free.}$

\[
\min J \implies x_0 \to 0.
\]

\[H = |u| + \lambda |u| \quad \text{opt} = \begin{cases} 
-1 & \text{if } \lambda > 1 \\
0 & \text{if } \lambda \in (-1, 1) \\
1 & \text{if } \lambda < -1 \\
? & \text{if } \lambda = \pm 1
\end{cases}
\]

\[-x = \frac{dH}{dx} = 0 \quad \sqrt{1+\lambda}
\]

$t_f$ is free, the system and objective are autonomous

$\implies H \text{opt} = 0 \implies \lambda = \pm 1$ (singular interval)

Notice: $A = 0 \implies$ singular intervals may exist.

Indeed, the solution is not unique. For any $t_f$,

\[u = -\frac{x_0}{t_f} \text{ satisfies the B.C. and } J = \int_{0}^{t_f} \frac{|x_0|}{t_f} \, dt = |x_0|
\]

(all have the same, optimal cost.)
Pr. 7.8  \( \bar{x} = \begin{pmatrix} 0 & 1 \\ 0 & -a \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \end{pmatrix} u \) \( a > 0 \)
\( l_u \leq 1 \)

\[ J = \int_0^{t_f} \beta + l_u \, dt \text{ free, } \beta > 0 \]
\( x_0 \rightarrow 0 \)

(Weighted min-time - min fuel problem)

\[ H = \beta + l_u + x^T (Ax + Bu) \]

\[ u_{opt} = \begin{cases} -1 & \text{if } \lambda_2 > 1 \\ 0 & \text{if } \lambda_2 \in (-1, 1) \\ 1 & \text{if } \lambda_2 < -1 \\ ? & \text{if } \lambda_2 = \pm 1 \end{cases} \]

\( \Rightarrow \lambda_1 = 0 \quad \lambda_2 = -\lambda_1 + a \lambda_2 \Rightarrow \lambda(t) = \begin{pmatrix} \eta_{10} \\ e^{at} \eta_{20} + a \eta_{10} / (1 - e^{at}) \end{pmatrix} \]

Depending on the sign and magnitude of the initial costate, we may have up to two switchings \((-1 \rightarrow 0 \rightarrow 1 \text{ or vice versa})\). There can be no singular intervals.

Critical values for \( \lambda, x \) at the switching times can be obtained from \( H = 0 \) e.g. switching

\(-1 \rightarrow 0 : \lambda_2 = 1 \) @ \( t^- \) \( u = -1 \Rightarrow \beta + 1 + (\lambda_{10}, 1)'(Ax + t_{10}) \)

@ \( t^+ \) \( u = 0 \Rightarrow \beta + (\lambda_{10}, 1)'Ax = 0 \)

etc.

Alternatively, we can sketch the switching surface to \( 0 \) with inputs \( \pm 1 \), and the 3 possible trajectories from \( x_0 \) (i.e. \( u = +1, -1, 0 \)). Note: for \( u = 0 \), \( x(t) = c - x_2(t) / a \), a straight line.
The optimal trajectory ($\beta \to \infty$) switches $\pm 1 \to \mp 1$ (red-green in our plot). The weighted time-fuel optimal connects the two segments with a straight line of slope $-\alpha$. The length of the segment depends on $\beta$ (but not straightforward to compute).
Pr 7.9
\[ x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \quad |u| \leq 1 \]
\[ x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow x_f = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad t_f = 4 \]
\[ J = \frac{1}{2} \int_0^4 u^2 \]

\[ H = \frac{1}{2} u^2 + \lambda^T (Ax + Bu) \]

\[ u_{opt} : \quad 2u + B^T \lambda = 0 \quad \Rightarrow \quad u = -\frac{B}{2} B^T \lambda \quad \text{(unconstr.)} \]

when \(|u| \leq 1 = u_{opt} = \text{sat} \left[ -\frac{2}{2} B^T \lambda \right] \rightarrow \]

\[ \frac{d}{dt} x = -2A^T x = -A^T x \quad \Rightarrow \quad \lambda_1 = c_1 \quad \Rightarrow \lambda = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

\[ x_2 = c_2 - \lambda_2 t \]

The unconstrained solution is given by
\[ u_{opt} = -\phi_2(t) \quad \text{and} \]
\[ \begin{pmatrix} 0 \\ x_f \end{pmatrix} = (A \quad -BB^T)(x) \quad \Rightarrow \quad (x(t)) = e^{-At}(x_0) \]
\[ \text{with} \quad x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad x_f = 0 \]
\[ \Rightarrow x_0 = -\phi^{-1}_{12}(4) \phi_{11} x_0 = \begin{pmatrix} 0.5625 \\ 1.3750 \end{pmatrix} \]

\[ x_2(t) = (t, -1) \lambda_0 \rightarrow \phi_{12}(4) x_0 \]

\[ \text{violates the bound} \]

\[ \Rightarrow \text{the soln. is not unconstrained} \Rightarrow \text{it changes entirely. Without solving the TPBVP, we can observe that the soln. will be equal to the unconstrained (} u_{opt} = -\lambda_2 \text{) after some time } t_1 \text{ (if the problem is feasible)}. \]
Then, in \((0, t_1)\), \((x^0, x^1) = (A - AT)(x^0, x^1) + (B_0)(-1)\) and in \((t_1, 4)\), \(x(t) = \hat{H}(x)\) with continuity at \(t_1\) being the boundary condition.

In the interval \((0, t_1)\), 
\[
x(t) = \begin{pmatrix} x_{10} - \frac{1}{2}t^2 + x_{20}t \\ x_{20} - t \end{pmatrix}
\]
where \(x_{10} = x_{20} = 1\).

In the interval \((t_1, 4)\), 
\[
\begin{pmatrix} x(t) \end{pmatrix} = e^{-H(t-t_1)} \begin{pmatrix} x(t_1) \end{pmatrix}
\]
\[\Rightarrow x(t_1) = - \phi_{12}^{-1}(4-t_1) \phi_{11}(4-t_1) x(t_1)\]
\[\Rightarrow u(t) = - \lambda_2(t) = \begin{pmatrix} (t-t_1), -1 \end{pmatrix} \phi_{12}^{-1}(4-t_1) \phi_{11}(4-t_1) \begin{pmatrix} 1-t^{\frac{3}{2}}+t_1^2 \\ 1-t_1 \end{pmatrix}\]

It is now a 1-parameter search to find \(t_1\) s.t. \(|u(t_1)| \leq 1\). The first (and only) time is at \(t_1 = 1\). At that time \(x(t_1) = \begin{pmatrix} 3/2 \\ 0 \end{pmatrix}\)

we compute & sketch the rest of the solution to check if \(|u(t)| \leq 1\) \(\forall t\).

The optimal cost is \(J_* = 1\) (analytically or numerically).
\[ x = (0 \ 1) x + (0 \ 1) u \]

\[ J = \int 1 \]

\[ |u| \leq 1, \ |x_2| \leq 2 \quad \text{(Velocity constraint)} \]

Let \( g(x_2-2) \) be such that \[ g \]

Define the new state \( \bar{x} = g^2 \) and the iso-perimeteric constraint \( \bar{x}(0) = 0, \bar{x}(t_f) = 0 \).

Then, the Hamiltonian is

\[ H = 1 + x^T (Ax + Bu) + \mu g^2 \leq \lambda, \mu : \text{costates} \]

Optimality:

\[ u^* = -\text{sign} (B^T \lambda) \]

\[ \bar{x} = -\frac{\partial H}{\partial x} = -A^T \lambda + 2\mu g (1) \]

\[ \bar{\mu} = 0 \]

When \( g \equiv 0 \), so that the solution satisfies the constraints, \( \bar{x} = -A^T \lambda \Rightarrow u^* \) is the time optimal input. This solution is Bang-Bang and may violate the velocity constraint. Intuitively, the optimal solution will use the \( \pm 1 \) controls when \( |x_2| < 2 \) and \( 0 \) control (constant velocity) when \( x_2 = 2 \).
The way this solution arises from our equation is by considering some time where $x_2 > 0$, but small. Then for large $\mu$, $x_2$ will change sign very quickly. Thus, the solution will chatter around the constraint, which is expected since the extremals still have $u = \pm 1$. In an average sense, the trajectory moves in the correct direction (intuitively the same as $u = 0$), see Fillipou for this. Notice that any $x_2 > 0$ will make $z(t_f) > 0$. But for $\mu \to \infty$, $z(t_f) \to 0$. 
### Problem 6.3

Build the table with entries:
- **Stage**, **State** & **Optimal Cost-to-go** ($J_*(x)$),
- **State** & **Optimal Control** ($u(x)$).

Starting from the last stage we get:

<table>
<thead>
<tr>
<th>STAGE</th>
<th>STATE</th>
<th>$J_*(x)$</th>
<th>$u(x)$</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>Z</td>
<td>0</td>
<td>N-N</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>H</td>
<td>3</td>
<td>H-N</td>
<td></td>
</tr>
<tr>
<td></td>
<td>I</td>
<td>4</td>
<td>I-N</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>E</td>
<td>10</td>
<td>E-H</td>
<td>$J_<em>(F) = \min \left{ \frac{J_</em>(E)}{J_*(I)} + 3 \right} = 6$</td>
</tr>
<tr>
<td></td>
<td>F</td>
<td>6</td>
<td>${F-H}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>G</td>
<td>9</td>
<td>G-I</td>
<td>Multiple minima.</td>
</tr>
<tr>
<td>2</td>
<td>C</td>
<td>12</td>
<td>C-F</td>
<td>$J_<em>(C) = \min \left{ \frac{J_</em>(E)}{J_*(F)} + 6 \right} = \min(12)$</td>
</tr>
<tr>
<td></td>
<td>D</td>
<td>13</td>
<td>D-F</td>
<td>$J_<em>(D) = \min \left{ \frac{J_</em>(C)}{J_*(F)} + 7 \right}$</td>
</tr>
<tr>
<td>1</td>
<td>L</td>
<td>15</td>
<td>${L-C }$</td>
<td>$J_<em>(L) = \min \left{ \frac{J_</em>(C)}{J_*(D)} + 3 \right}$</td>
</tr>
</tbody>
</table>

Minimum Cost: $L-N = 15$

Optimal paths: $\{L-C-F-H-N, L-C-F-I-N, L-D-F-H-N, L-D-F-I-N\}$ (just go through $F$)