Optimal Control Problems

April 7, 2006

1

1.1 Max-perimeter rectangle inscribed in an ellipse

Find x, y to maximize $J = 4(x + y)$ subject to the constraint $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Evaluate for $a = 1,\ b = 2$.

1.2 Max-volume parallelepiped inscribed in an ellipsoid

Find x, y, z to maximize $J = 8(xyz)$ subject to the constraint $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Evaluate for $a = 1,\ b = 2,\ c = 3$.

1.3 Min-distance between two lines

A straight line in 3-D is the intersection of two planes, represented by $Ax = b$ where A is a 2×3 matrix. Use constrained optimization with Lagrange multipliers to show that the points on the lines closest to each other are

$$x^1 = D_1^{-1}(d_1 + d_2 - C_1 d_2), \quad x^2 = D_2^{-1}(d_1 + d_2 - C_2 d_1)$$

where,

$$\begin{array}{lll} C_1 = A_1^\top (A_1 A_1^\top)^{-1} A_1, & d_1 = A_1^\top (A_1 A_1^\top)^{-1} b_1, & D_1 = C_1 + C_2 - C_1 C_2 \\ C_2 = A_2^\top (A_2 A_2^\top)^{-1} A_2, & d_2 = A_2^\top (A_2 A_2^\top)^{-1} b_2, & D_2 = C_1 + C_2 - C_2 C_1 \end{array}$$

1.4 Min-time path

A lifeguard can run at velocity v_1 and swim at velocity v_2. Let $y > y_1$ be the water and $y < y_1$ be the land. The lifeguard is initially at the origin. Find the min-time path to (x_2, y_2). Assume that the optimal path consists of two straight lines, so the problem can be stated as

$$\begin{aligned} \min_{\theta_1, \theta_2} L &= \frac{y_1 \sec\theta_1}{v_1} + \frac{(y_2 - y_1)\sec\theta_2}{v_2} \\ \text{s.t.} \quad & x_2 - y_1 \tan\theta_1 - (y_2 - y_1)\tan\theta_2 = 0 \end{aligned}$$

a. Use a Lagrange multiplier to show that $\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$, which is Snell's law.
b. Snell's law and the constraint are the two nonlinear equations in the unknowns. Evaluate the solution numerically for $y_1 = 100$, $x_2 = y_2 = 300$, $v_1 = 25$, $v_2 = 6$ (min-time = 45.1, straight-line-time = 52.8).

1.5 Quadratic index with linear equality constraints

Find the parameter vector x to minimize $J = \frac{1}{2}(x^\top Q x + u^\top R u)$ subject to $x + Gu = c$. Using Lagrange multipliers (λ) show that the solution is

$$u = kC, \quad L_{min} = \frac{1}{2} c^\top S c, \quad -\lambda = Sc = \left[\frac{\partial L_{min}}{\partial c}\right]$$

where,

$$K = (R + G^\top QG)^{-1}G^\top Q = R^{-1}G^\top S$$
$$S = Q - QG(R + G^\top QG)^{-1}G^\top Q = (Q^{-1} + GR^{-1}G^\top)^{-1}$$

The alternative expressions are valid when R, Q are invertible and are a special case of the *matrix inversion lemma*.

1.6 Max singular values of a matrix

Find the maximum and minimum values of the magnitude of the vector $y = Ax$ subject to the constraint that x has unit magnitude.

1.7 Min distance between ellipses

Develop an algorithm to find the minimum distance between two ellipses $(x - c_1)^\top P_1(x - c_1) = 1$ and $(x - c_2)^\top P_2(x - c_2) = 1$. Evaluate for $c_1 = 0$, $P_1 = I$, $c_2 = [10; 10]$, $P_2 = \text{diag}(1, 5)$.

2

2.1 The brachistochrone problem

A bead slides on a wire without friction in a gravitational field. Find the shape of the wire to minimize the time it takes to cover a given horizontal distance x_f.

Posed and solved by Jakob Bernoulli in the 17th century.

2.2 *Zermelo's problem

A ship travels with constant velocity V with respect to the water through a region where the velocity of the current is parallel to the x-axis and varies with y so that the ship movement satisfies

$$\dot{x} = V \cos\theta + u_c(y), \quad \dot{y} = V \sin\theta$$

where θ is the ship's heading relative to the x-axis. Find $\theta(t)$ to minimize the time to move from the origin to a given final position (x_f, y_f) (if feasible).

2.3 Leaky reservoir

The differential equation that describes a leaky reservoir is

$$\dot{x} = -0.1x + u$$

where x is the height of water and u is the net inflow rate of water. Assume that $0 \leq u(t) \leq M$ for all t.

1. Find the optimal control minimizing
$$J = \int_0^{100} -x(t)dt$$
2. Repeat part 1 with the additional constraint
$$\int_0^{100} u(t)dt = K(\text{ a known constant})$$
3. Find the optimal control minimizing $J = -x(100)$ subject to
$$\int_0^{100} u(t)dt = K$$

2.4 Shape of a hanging chain

Given a chain of length $2L$ hanging from two points at the same level and at distance 2ℓ apart, find the shape of a hanging chain $y(x)$ that minimizes the potential energy in the earth's gravitational field. That is, find u to minimize $J = -g\sigma \int_{-\ell}^{\ell} y ds$ subject to $\int_{-\ell}^{\ell} ds = 2L$, $y(-\ell) = y(\ell) = 0$, g is the gravitational constant, σ is the mass per unit length, $ds = \sqrt{1+u^2}dx$ and $dy/dx = u$. Show that the curve is the catenary

$$y = H\left(\cosh\frac{x}{H} - \cosh\frac{\ell}{H}\right)$$

where H is determined by $L/H = \sinh \ell/H$.

Posed by Galileo in 1638, solved by Jakob and Johannes Bernoulli and Leibnitz in 1690-1692.

2.5 Min surface area of soap film

Find the shape of a soap film stretched between two coaxial loops, of radius a and at distance 2ℓ apart. The surface is such that it has minimum area and, because of the problem symmetry, it is given by

$$A = 2\pi \int_{-\ell}^{\ell} r\sqrt{1+u^2}dx$$

where, (r, x) are the cylindrical coordinates, $\frac{dr}{dx} = u$, $r(-\ell) = r(\ell) = a$.

The surface is the catenary $r = H\cosh\frac{x}{H}$, where ℓ/H is determined by $a\ell(\ell/H) = \cosh \ell/H$. This equation has two solutions for $0 \le \ell/a \le 0.663$ and no solution for $\ell/a > 0.663$.

The minimum area is

$$A_{\min} = \begin{cases} 2\pi a^2[\tanh(\ell/H) + (\ell/H)\mathrm{sech}^2(\ell/H) & \text{if } 0 \le \ell/a \le 0.528 \\ 2\pi a^2 & \text{if } \ell/a \ge 0.528 \end{cases}$$

3

3.1

For the optimal control problem

$$\begin{aligned} \min_{x,u} J(x,u) &= \frac{1}{2}\int_0^1 (2x_1^2 + x_2^2 + u^2)\, dt \\ \text{s.t.} \quad & \dot{x}_1 = x_2 \\ & \dot{x}_2 = -x_1 + [1 - x_1^2]x_2 + u \end{aligned}$$

with $x(0) = x_0$, $x(t_f) = x_f$ specified, determine

1. the costate equations;
2. the optimal control (minimizing the Hamiltonian) when u is not constrained.

3.2

The system

$$\dot{x} = -x + u$$

is to be transferred to the origin from an arbitrary initial state in 1 unit of time. Determine the control that achieves this objective and minimizes

$$\min_{x,u} J(x,u) = \frac{1}{2}\int_0^1 (3x^2 + u^2)\, dt$$

There are no constraints on u.

3.3 LQR Stabilization

Consider the LQR optimal control problem

$$
\begin{aligned}
\min J &= \frac{1}{2}\int_0^\infty (x^\top Q x + u^\top R u) dt \\
\text{s.t.} \quad & \dot{x} = Ax + Bu
\end{aligned}
$$

where $R = R^\top > 0$, $Q = C^\top C$, (A, B, C) minimal (c.c. and c.o.). The optimal control is $u = Kx$ where $K = -R^{-1}B^\top P$ is defined in terms of the Riccati solution P.

1. Show that this controller stabilizes the system. (Use $V = x^\top P x$ as a Lyapunov function and compute its time derivative along the trajectories of x.)

2. Consider the controller $u = \rho K x$. With the same Lyapunov approach, find the minimum and maximum values of ρ for which closed-loop stability is guaranteed. Interpret your results in terms of an upward and downward gain margin.

3.4 Exponentially weighted LQR

Consider the LQR optimal control problem

$$
\begin{aligned}
\min J &= \frac{1}{2}\int_0^\infty e^{2\delta t}(x^\top Q x + u^\top R u) dt \\
\text{s.t.} \quad & \dot{x} = Ax + Bu
\end{aligned}
$$

where $R = R^\top > 0$, $Q = C^\top C$, (A, B, C) minimal (c.c. and c.o.) and $\delta > 0$. Compute the optimal control in a feedback form ($u = Kx$) and show that this controller provides a stability margin of at least δ. That is, $Re\text{eig}(A + BK) < -\delta$ and the states converge to zero at least as fast as $ce^{-\delta t}$ where c is a constant.

4

4.1 Time-optimal control

For the system

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u
\end{aligned}
$$

determine the optimal control that transfers an initial state to the state $x_f = [1, 1]^\top$ in minimum time, subject to the constraint $|u(t)| \le 1$. In particular, develop expressions for the switching curve and give the optimal control in a feedback form.

4.2 Weighted time-energy-optimal control

Determine the necessary conditions that should be satisfied by the extremals of

$$
\begin{aligned}
\min_{x,u} J(x, u) &= \int_0^{t_f} (\alpha + u^2)\, dt \\
\text{s.t.} \quad & \dot{x}_1 = x_2 \\
& \dot{x}_2 = u
\end{aligned}
$$

where $\alpha > 0$ is a constant and $x(0) = x_0$, $x(t_f) = x_f$ are given.

5

5.1 Aircraft min-time to climb [Bryson et al. ASME J. Appl. Mech., 1962]

Find the minimum time flight path for the F4 A/C to climb to 20km arriving with Mach 1 and level flight.

Using a point-mass approximation (good for long-term performance problems) the equations of motion are:

$$\begin{aligned} m\dot{V} &= T\cos(\alpha+\epsilon) - D - mg\sin\gamma \\ mV\dot{\gamma} &= T\sin(\alpha+\epsilon) + L - mg\cos\gamma \\ \dot{h} &= V\sin\gamma \\ \dot{m} &= -\frac{T}{cg} \\ \dot{x} &= V\cos\gamma \end{aligned}$$

where, V is velocity, γ is flight path angle, h is altitude, m is mass, x is horizontal range, T is thrust, D is drag, L is lift, α is angle of attack, ϵ is thrust angle relative to the zero lift axis (constant, usually a few degrees), g is the gravitational constant, c is a fuel consumption coefficient equal to 1600 sec. The angle of attack is used as the control variable.

The manufacturer's data for lift, drag, and max thrust were used after an analytical fit to allow the easy computation of partial derivatives.

$$\begin{aligned} T_{\max} &= [1 M M^2 M^3 M^4] Q [1; h; h^2; h^3; h^4] \\ Q &= \begin{bmatrix} 30.21 & -0.668 & -6.877 & 1.951 & -0.1512 \\ -33.80 & 3.347 & 18.13 & -5.865 & 0.4757 \\ 100.8 & -77.56 & 5.441 & 2.864 & -0.3355 \\ -78.99 & 101.4 & -30.28 & 3.236 & -0.1089 \\ 18.74 & -31.60 & 12.04 & -1.785 & 0.09417 \end{bmatrix} \end{aligned}$$

where, $T_{\max}$ is in units of 1000lb, M is the Mach number, h is altitude in units of 10kft.

The aerodynamic data of the F4 A/C are:

Drag coefficient: $C_D = C_{Do} + \kappa C_{L\alpha}\alpha^2$

Lift Coefficient: $C_L = C_{L\alpha}\alpha$

The analytical fits for the various parameters are given as follows:

$$M < 1.15 \begin{cases} C_{Do} &= 0.013 + 0.0144\left[1 + \tanh\left(\frac{M-0.98}{0.06}\right)\right] \\ C_{L\alpha} &= 3.44 + \frac{1}{\cosh^2\left(\frac{M-1}{0.06}\right)} \\ \kappa &= 0.54 + 0.15\left[1 + \tanh\left(\frac{M-0.9}{0.06}\right)\right] \end{cases}$$

$$M > 1.15 \begin{cases} C_{Do} &= 0.013 + 0.0144\left[1 + \tanh\left(\frac{0.17}{0.06}\right)\right] - 0.011(M-1.15) \\ C_{L\alpha} &= 3.44 + \frac{1}{\cosh^2\left(\frac{0.15}{0.06}\right)} - \frac{0.96}{0.63}(M-1.15) \\ \kappa &= 0.54 + 0.15\left[1 + \tanh\left(\frac{0.25}{0.06}\right)\right] + 0.14(M-1.15) \end{cases}$$

5.2 Furnace temperature ramp-up problem [Tsakalis and Stoddard, 6th IEEE Intl. Conf. ETFA, Los Angeles, 1997]

The optimal ramp-up problem arises frequently in applications where the system undergoes steady-state transitions subject to bounded controls. The motivation of the particular problem studied here is from the temperature control of diffusion furnaces used in semiconductor manufacturing.

The temperature control problem in diffusion furnaces is to follow a desired temperature trajectory uniformly along the length of the tube. The control input is the power supplied to heating coils surrounding the tube. Each coil represents a so-called heating zone; industrial diffusion furnaces may contain 3 to 9 heating zones. Temperature measurements are obtained by means of thermocouples at different locations along the tube, roughly corresponding to the center of each heating zone.

In a typical operation, the furnace is idling at a low temperature (300-600 deg.C). After loading the wafers to be processed, the temperature is raised to the processing specifications (500-1100 deg.C) and is maintained constant throughout the wafer processing (e.g., oxidation, annealing, etc). After the end of the process, the temperature is ramped down to the idling conditions until the next cycle.

The main objective of the controller is to maintain spatially uniform, accurate, and repeatable processing temperature in the presence of perturbations, such as varying gas flows, heat losses etc. However, it should also provide good control during the ramp-up/ramp-down operations so that all wafers face similar conditions. Furthermore, in order to reduce the cycle-time and increase productivity, the steady-state transitions should be performed as quickly as possible but in a controlled manner.

While accurate furnace models are infinite dimensional and (mildly) nonlinear, linear low-order approximations can be obtained via system identification techniques from data around an operating point. The accuracy of such models is adequate in a 50-100 deg. neighborhood of the identification conditions and this has proven sufficient for control design purposes.

With this background, the objective of this assignment is to investigate the properties of trajectories that optimize the transition between two steady-states. Even though the system model represents an actual furnace, several simplifications are made in the interest of programming ease and computational speed. In particular, the controller can use full-state information and the system is treated as linear in the region of operation. The main objective is to minimize the time to raise the operating temperature from idling to processing conditions while maintaining temperature uniformity across the furnace. In addition to that, control input saturation presents an important constraint that must be taken into account.

Problem Statement: Consider a square linear system with problem data [A,B,C,D] supplied below. Design a control input and find the optimal state trajectory that minimizes the time to transition from the steady-state $y = 0$, $\dot{x} = 0$, to the steady-state $y = [100; 100; 100]$, $\dot{x} = 0$, subject to the constraints:
1. $-0.2 \leq u_i \leq 0.8$, for all i 2. $|y_i - y_j| \leq 0.5$deg, for all i,j, where, u_i is the i-th input (fraction of supplied power) and y_i is the i-th temperature.

To enforce the constraints on the input or the states, one approach that fits the Mayer formulation is to define new states ($\dot{z} = g^2(u)$ or $\dot{z} = g^2(x)$) with boundary conditions $z(0) = 0$, $z(t_f) = 0$, where $g(.)$ is a function that is zero when the constraints are satisfied and nonzero when they are violated. The square serves to ensure that $\dot{z}$ is differentiable, for the purposes of computing gradients.

Alternatively, a penalty approach may be used to enforce approximately the constraints. That is, the integral cost function is augmented with a term $Kg^2(x, u)$, where K is a constant that determines the penalty for violating the constraint. Here, the constraints can be considered as "soft." A violation of the power fraction by 0.01, or the output matching by 0.1 are acceptable. Obviously, larger K will force the solution to stay closer to the constraints, but they tend to make the problem ill-conditioned and slow down the optimization.

After computing the optimal solution, it is interesting to compare it with the results of an ad-hoc solution using an LQR tracking controller with a ramp reference input. For this, consider the i-th reference as $r_i(t) = c_i ramp(t-t_i)$ and adjust the ramp rate c_i and the time shift t_i to avoid violation of the constraints, as much as possible. For the simulations, the input constraint should be enforced using saturation functions, regardless of the value of the computed input. For controllers that converge to the steady-state asymptotically, the convergence time can be taken as the maximum time when all the outputs enter the envelope $|y_i - 100| < 0.2$ deg.

Furnace Problem Data (This model corresponds to a steady-state of all outputs being 900 deg C, and the input powers being approximately 0.2 (20%). Time is measured in minutes.)

```
A1=[ -8.3342e+000   -1.3856e+001   2.9473e+000
      8.8432e-001    -2.3043e-015    4.5864e+000
      2.3438e-002      1.0633e-003   -8.2887e-003 ]

A2=[ -1.2461e+001     -1.3856e+001   4.4069e+000
      1.8471e+000    -2.8167e-015    4.2459e+000
```

```
      4.2108e-002      1.0633e-003    -1.4891e-002 ]

A3=[ -8.1379e+000     -1.3856e+001     2.8779e+000
      1.2471e+000    -2.4974e-015     4.4581e+000
      4.8808e-002       1.0633e-003    -1.7260e-002 ]

B=[ -1.2818e+000     -9.0649e-002      -1.6503e-002
    -1.8955e+000     -1.1612e-001        1.2613e-002
    -2.1570e+000     -7.2264e-002        4.4509e-002
     1.5346e-002     -8.4137e-001       -2.4555e-002
     3.6399e-003     -5.0429e-001       -1.6513e-002
    -1.8155e-002     -1.6804e+000       -2.2522e-002
     6.6923e-003      -5.8721e-003       -1.4088e+000
     8.8916e-003      -1.0094e-001       -1.5344e+000
     2.4779e-002      -1.0204e-001       -2.2465e+000 ]

C1=[ 6.5318e+000     -3.4766e-015     -2.3099e+000 ]

A = [A1, 0*A1, 0*A1; 0*A1, A2, 0*A1; 0*A1, 0*A1, A3]

B = B

C = [C1, 0*C1, 0*C1; 0*C1, C1, 0*C1; 0*C1, 0*C1, C1]

D = zeros(3,3)

save isoc A B C D
```

Ref: A. Bryson Jr., Dynamic Optimization. Addison-Wesley, 1999.

Solutions to Optimal Control Problems

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1

1.1 Max-perimeter rectangle inscribed in an ellipse

Adjoin the constraint to the performance index with a Lagrange multiplier λ:

$$\bar{P} = 4(x+y) + \lambda(x^2/a^2 + y^2/b^2 - 1) \ . \tag{1}$$

The stationarity conditions are then

$$\bar{P}_x = 4 + 2\lambda x/a^2 = 0 \ , \tag{2}$$
$$\bar{P}_y = 4 + 2\lambda y/b^2 = 0 \ . \tag{3}$$

(2), (3), and the constraint are three equations for $x,\ y,\ \lambda$. Eliminating λ between (2) and (3) gives

$$\lambda = -2a^2/x = -2b^2/y \ \Rightarrow \ x/a^2 = y/b^2 \ . \tag{4}$$

Substituting (4) into the constraint gives

$$(a^2y/b^2)^2/a^2 + y^2/b^2 = 1 \ \Rightarrow \ y^2 = b^4/(a^2+b^2) \ \Rightarrow x^2 = a^4/(a^2+b^2) \ . \tag{5}$$

1.2 Max-volume parallelepiped inscribed in an ellipsoid

Adjoin the constraint to the performance index with a Lagrange multiplier λ:

$$\bar{V} = 8xyz + \lambda(x^2/a^2 + y^2/b^2 + z^2/c^2 - 1) \ . \tag{1}$$

The stationarity conditions are then

$$\bar{V}_x = 8yz + 2\lambda x/a^2 = 0 \ , \tag{2}$$
$$\bar{V}_y = 8xz + 2\lambda y/b^2 = 0 \ , \tag{3}$$
$$\bar{V}_z = 8xy + 2\lambda z/c^2 = 0 \ . \tag{4}$$

(2)-(4), and the constraint are four equations for $x,\ y,\ ,z,\ \lambda$. Eliminating λ between (2) and (3) gives

$$\lambda = -4a^2yz/x = -4b^2xz/y \ , \ \Rightarrow \ x^2/a^2 = y^2/b^2 \ . \tag{5}$$

Similarly, eliminating λ between (2) and (4) gives

$$x^2/a^2 = z^2/c^2 \ . \tag{6}$$

Substituting into the constraint gives

$$3x^2/a^2 = 1 \ \Rightarrow x^2/a^2 = y^2/b^2 = z^2/c^2 = 1/3 \ . \tag{7}$$

1.3 Min-distance between two lines

The problem may be stated as finding $(x_1,\ x_2)$ to min $L = (x_1 - x_2)^T(x_1 - x_2)/2$ subject to $A_1x_1 = b_1$ and $A_2x_2 = b_2$ where $(A_1,\ A_2)$ each have two rows and three columns.

Adjoin the constraints with Lagrange multiplier vectors λ_1 and λ_2

$$H \triangleq L + \lambda_1^T(A_1x_1 - b_1) + \lambda_2^T(A_2x_2 - b_2) \ . \tag{1}$$

The stationarity conditions are

$$\begin{aligned} 0 &= H_{x_1} = (x_1 - x_2)^T + \lambda_1^T A_1 \ , & (2)\\ 0 &= H_{x_2} = -(x_1 - x_2)^T + \lambda_2^T A_2 \ . & (3) \end{aligned}$$

Equations (2), (3), and the constraints are 3+3+2+2=10 linear equations for the 3+3+2+2=10 unknowns $x_1,\ x_2,\ \lambda_1,\ \lambda_2$. Eliminating x_1 using (2) and the first constraint gives

$$\lambda_1 = (A_1A_1^T)^{-1}(A_1x_2 - b_1) \ . \tag{4}$$

Eliminating x_2 using (3) and the second constraint set gives

$$\lambda_2 = (A_2A_2^T)^{-1}(A_2x_1 - b_2) \ . \tag{5}$$

Substituting (4) and (4) into the transposes of (2) and (3) and solving for x_1 and x_2 gives the stated results.

1.4 Min-time path

a. Form the $H-$ function

$$H = y_1 \sec\theta_1/v_1 + (y_2 - y_1)\sec\theta_2/v_2 + \lambda[x_2 - y_1\tan\theta_1 - (y_2 - y_1)\tan\theta_2] \ . \tag{1}$$

Necessary conditions for a stationary solution are

$$\begin{aligned} 0 = H_{\theta_1} &= y_1\sec\theta_1\tan\theta_1/v_1 - \lambda\, y_1\sec^2\theta_1 \ , & (2)\\ 0 = H_{\theta_2} &= (y_2 - y_1)\sec\theta_2\tan\theta_2/v_2 - \lambda\,(y_2 - y_1)\sec^2\theta_2 \ . & (3) \end{aligned}$$

Eliminating λ between (2) and (3) gives

$$\lambda = \sin\theta_1/v_1 = \sin\theta_2/v_2 \ , \quad \text{Q. E. D.} \tag{4}$$

b. Eqn. (4) and the constraint eqn. are two nonlinear eqns. for $(\theta_1,\ \theta_2)$. They are solved using the MATLAB Optimization Toolbox command FSOLVE.

An alternative computation is by implementing a Newton algorithm for the solution $(\theta(k+1) = \theta(k) - [\nabla F(\theta(k))]^{-1}F(\theta(k)))$, where:

$$F(\theta) = \begin{bmatrix} \sin(\theta_1)/v_1 - \sin(\theta_2)/v_2 \\ x_2 - y_1\tan(\theta_1) - (y_2 - y_1)\tan(\theta_2) \end{bmatrix} \ ; \quad \nabla F(\theta) = \begin{bmatrix} \cos(\theta_1)/v_1 & -\cos(\theta_2)/v_2 \\ -y_1/\cos^2(\theta_1) & (y_1 - y_2)/\cos^2(\theta_2) \end{bmatrix}$$

The iteration converges from initial conditions $[\pi/4, \pi/4]^T$ (the straight line path) or $[0, 0]^T$ in 20-30 iterations. The resulting angles are correct modulo 2π. (A quick fix of this is to use $\theta = \tan^{-1}(\tan(\theta))$ after each iteration.) However, checking the progress of the solution, we see a fairly "violent" behavior until the final stages of the iteration. In general, such a generic algorithm need not converge when started far from the solution.

1.5 Quadratic index with linear equality constraints

Adjoin the constraints with the Lagrange multiplier vector λ

$$H \stackrel{\Delta}{=} (x^T Q x + u^T R u)/2 + \lambda^T (x + Gu - c) \ . \tag{1}$$

It is assumed throughout that Q, R are symmetric. Also, for the existence of a minimizer, the second order condition (hessian positive semi-definite) yields $R + G^T Q G \geq 0$ (its positive definiteness is a sufficient condition). The stationarity conditions are

$$0 = H_x = x^T Q + \lambda^T \ , \tag{2}$$
$$0 = H_u = u^T R + \lambda^T G \ . \tag{3}$$

Equations (2), (3), and the constraints are linear equations for the unknowns $x, \ u, \ \lambda$. Solving (2) for λ gives

$$\lambda = -Qx \equiv -Q(c - Gu) \ . \tag{4}$$

Substituting (4) in (3) and solving for u gives

$$u = (R + G^T Q G)^{-1} G^T Q c = Kc \ , \tag{5}$$

Substituting (5) into the constraint gives

$$x = [I - G(R + G^T Q G)^{-1} G^T Q] c \ . \tag{6}$$

Substituting (6) into (4) gives

$$-\lambda = Q[I - G(R + G^T Q G)^{-1} G^T Q] c = Sc \tag{7}$$

Notice that, up to this point, the solution does not require Q, R to be invertible (only $R + G^T Q G$ should be invertible).

To derive the rest of the expressions, assume that R is invertible and combine (3) and (4) to get

$$u = R^{-1} G^T Q x \tag{8}$$

Since $Qx = -\lambda = Sc$, this establishes the second expression for K. Furthermore, substituting (8) in the constraint and solving for x, we get

$$x = [I + G R^{-1} G^T Q]^{-1} c = Q^{-1} [Q^{-1} + G R^{-1} G^T]^{-1} c \tag{9}$$

where we assumed the invertibility of Q. Comparing (9) with (7), the second expression for S follows (the equalities must hold for all c). These identities may also be derived by direct verification.

Finally, it remains to show that L^{min} satisfies the given expression. For this, at the optimum $x = Q^{-1} S c$ and $u = Kc$ so,

$$L^{min} = c^T [S^T Q^{-1} Q Q^{-1} S + K^T R K] c/2 \tag{10}$$

Substituting the second expression for K and noting that $S = S^T$, we have

$$L^{min} = c^T [S Q^{-1} S + S G R^{-1} R R^{-1} G^T S] c/2 = c^T S [Q^{-1} + G R^{-1} R R^{-1} G^T] S c/2 = c^T S S^{-1} S c/2 = c^T S c/2 \tag{11}$$

NOTE: The expressions for S are a special case of the *matrix inversion lemma*

$$(A + BCD)^{-1} = A^{-1} - A^{-1} B (C + D A^{-1} B)^{-1} D A^{-1} \ . \tag{12}$$

The significance of this lemma is the following: If A^{-1} is already known, the inverse of $(A + BCD)$ can be calculated using the RHS of (15) which requires inverting a lower order matrix if the dimension of C is less than the dimension of A.

1.6 Max singular values of a matrix

Form $H \stackrel{\Delta}{=} x^T A^T Ax/2 + \lambda(1 - x^T x)$. A necessary condition for a min is that

$$0 = H_x = x^T A^T A - \lambda x \quad \Rightarrow A^T Ax = \lambda x \ . \tag{1}$$

This means that λ is an eigenvalue and x is an eigenvector of $A^T A$. From the second order necessary conditions for a min, $H_x x \geq 0$, we have that $A^T A - \lambda I \geq 0$ so λ must be the smallest eigenvalue of $A^T A$. (Similarly for a max where λ is the largest eigenvalue of $A^T A$.)

Multiplying (1) by x^T gives

$$x^T A^T Ax = x^T \lambda x \equiv \lambda x^T x \equiv \lambda \ , \tag{2}$$

so the (smallest, largest) eigenvalue of $A^T A$ is the square of (min, max) value of $|Ax|$, Q. E. D.

NOTE: The singular value decomposition (SVD) of a rectangular matrix A is

$$A = \sum_{i=1}^{q} \alpha_i u_i^T v_i \ , \tag{3}$$

where $v_i,\ u_i$ = are unit column eigenvectors of $(A^T A,\ AA^T)$ and α_i is the square root of the i^{th} singular value of $A^T A$, q is the min dimension of A. This decomposes the matrix into a sum of rank one matrices (outer products of unit vectors). See 'help SVD' in MATLAB.

1.7 Min distance between ellipses

This problem is formulated as

$$\min_{x_1, x_2 \in \mathbf{R}^2} (x_1 - x_2)^T (x_1 - x_2)/2$$
$$\text{s.t. } x_1^T P_1 x_1 = 1, \quad x_2^T P_2 x_2 = 1$$

where $P_1 = R^T(\theta)\text{diag}(1/4, 1)R(\theta)$ and $P_2 = \text{diag}(1/16, 1/4)$. Then, the H-function becomes

$$H = (x_1 - x_2)^T (x_1 - x_2)/2 + \lambda_1(x_1^T P_1 x_1 - 1) + \lambda_2(x_2^T P_2 x_2 - 1)$$

and the rest of the solution follows as usual (solve the 6 equations $H_{x_1} = 0$, $H_{x_2} = 0$ and the two constraints for $x_1, x_2, \lambda_1, \lambda_2$).

An interesting related problem is to compute the minimizer when the point x_2 is fixed. This problem finds applications in orthogonal projections of an exterior point onto an ellipsoid. While this is a convex problem and a local minimum is also a global minimum, the first order necessary conditions have more than one solution where a Newton algorithm may get stuck. In this case the second order conditions can serve to determine an interval for the Lagrange multiplier at the minimum and, thus, result in a very fast and reliable algorithm. (See notes on adaptive algorithms, Projection on an ellipsoid for details.) The code implementing this algorithm is given below:

```
function orppr1=orppr1(theta,Ra,cent,epsst);
%
% USAGE: orppr1(theta(n),Ra(n x n),cent(n),epsst)
%
% The function orppr1 computes the orthogonal projection of the parameter
% vector theta on the ellipsoid (x-cent)'Ra(x-cent)<=1 with
% tolerance epsst.
%

NN=length(theta);INN=eye(NN,NN);
epsst2=1+epsst;
xi=theta-cent; AA=(xi'*Ra*xi);
if AA < epsst2
```

```
   orppr1=theta;
else
   b=sqrt(AA);
   Z=(INN+b*Ra);
   z1=Z\xi;r1=Ra*z1;z2=Z\r1;
   f=xi'*z2-1;
   while abs(f) >= epsst
      r2=Ra*z2;
      grad=-2*xi'*(Z\r2);
      db=-f/grad;
      while (b+db) <= 0
         db=db/2;
      end
      b=b+db;
      Z=(INN+b*Ra);
      z1=Z\xi;r1=Ra*z1;z2=Z\r1;
      f=xi'*z2-1;
   end
   orppr1=z1+cent;
end
```

Unfortunately, projections on more general convex sets are not as straightforward.

2

2.1 The brachistochrone problem

Let $t_o, V_o, \ldots$ denote the various quantities in the original coordinates and define the transformations:

$$t = t_o\sqrt{g/x_f}, \quad V = V_o/\sqrt{gx_f}, \quad x = x_o/x_f, \quad y = y_o/x_f$$

Then, the equations of motion become

$$\dot{V} = \dot{V}_o/g = \sin\gamma, \quad \dot{x} = \dot{x}_o/\sqrt{gx_f} = V\cos\gamma, \quad \dot{x} = \dot{y}_o/\sqrt{gx_f} = V\sin\gamma$$

We immediately observe that $\dot{y} = V\dot{V}$ and, therefore, $y = \frac{1}{2}V^2$ where the integration constant is zero since $V(0) = y(0) = 0$. This eliminates y from the rest of the problem.

The optimal control problem can now be written as

$$\begin{aligned} \min_{\gamma,V,x} \quad & \int_0^{t_f} 1dt \\ \text{s.t.} \quad & \dot{V} = \sin\gamma \\ & \dot{x} = V\cos\gamma \\ & x(0) = 0\ , \quad x(t_f) = 1\ , \quad V(0) = 0 \end{aligned}$$

The Hamiltonian for this problem is

$$H = 1 + \lambda_1\sin\gamma + \lambda_2 V\cos\gamma$$

from which the necessary conditions for the optimal control are

$$\begin{aligned} & u: \ \frac{\partial H}{\partial\gamma} = 0 \Rightarrow \lambda_1\cos\gamma = \lambda_2 V\sin\gamma \\ & \dot{\lambda}_1 = -\lambda_2\cos\gamma \\ & \dot{\lambda}_2 = 0 \Rightarrow \lambda_2 = \text{const.} \end{aligned}$$

At $t = 0$, and using the free final time condition $H = 0$, we have that $\lambda_1(0) = -1/\sin\gamma(0) \neq 0$. So, from the minimization of the Hamiltonian w.r.t. γ it follows that $\cos\gamma(0) = 0 \Rightarrow \gamma(0) = \pi/2$. Eliminating λ_1 from $H = 0$ and $\lambda_1 \cos\gamma = \lambda_2 V \sin\gamma$ we obtain

$$-\lambda_2 V \cos^2\gamma - \cos\gamma = \lambda_2 V \sin^\gamma \Rightarrow \cos\gamma = -\lambda_2 V$$

Differentiation of the last equation yields

$$\lambda_2 \dot{V} = \sin(\gamma)\dot{\gamma} \Rightarrow \dot{\gamma} = \lambda_2 \Rightarrow \gamma(t) = \lambda_2 t + \pi/2$$

where the integration constant is determined from the previously computed $\gamma(0)$.

With $\gamma(t)$ available, V and x can now be integrated directly yielding

$$\begin{aligned} V(t) &= \frac{\sin\lambda_2 t}{\lambda_2} \\ x(t) &= -\frac{1}{2\lambda_2}\left[t - \frac{\sin 2\lambda_2 t}{2\lambda_2}\right] \end{aligned}$$

where we used the trigonometric identity $\sin^2 a = (1-\cos 2a)/2$ and the integration constants were determined from $V(0), x(0)$. Together with $y = \frac{1}{2}V^2$ and the definition $b = -\lambda_2$, these equations describe the optimal solution. The values of b and t_f are determined indirectly by solving $x(t_f) = 1,\ y(t_f) = y_f$.

With these expressions it is interesting to consider the problem of minimizing t_f to reach $x(t_f) = 1$ with $y(t_f)$ being free. Letting $a = 2bt_f$, we have that the optimal solution at t_f satisfies

$$\begin{aligned} \frac{a^2}{t_f^2} &= a - \sin a \\ \frac{a^2}{t_f^2} y_f &= 1 - \cos a \end{aligned}$$

Minimizing t_f^2 w.r.t. a from the first equation, we get that the minimizer satisfies

$$a_{opt} - 2\sin a_{opt} + a_{opt}\cos a_{opt} = 0$$

whose solution is $a_{opt} = \pi$. This, in turn, yields

$$t_{f,opt} = \sqrt{\pi}\ ,\quad y_{f,opt} = \frac{2}{\pi}\ ,\quad \left(b_{opt} = \frac{\sqrt{\pi}}{2}\right)$$

2.2 *Zermelo's problem

2.3 Leaky Reservoir

Part 1: $H = -x - 0.1\lambda x + \lambda u$ and the costate equation is

$$\dot{\lambda} = 1 + 0.1\lambda\,;\quad \lambda(t_f) = 0$$

where $t_f = 100$. Then, the optimal control is

$$u = \arg\min_u H = \begin{cases} M & \text{if } \lambda < 0 \\ 0 & \text{if } \lambda > 0 \end{cases}$$

Solving the costate equations,

$$\begin{aligned} \lambda(t) &= e^{0.1t}\lambda(0) + \int_0^t e^{0.1(t-\tau)}d\tau \\ &= -10 + (10 + \lambda(0))e^{0.1t} \end{aligned}$$

With $\lambda(0)$ easily computed from the terminal condition, $\lambda(0) = -10 + 10/e^{10}$, we have that $\lambda(t) < 0$ for $t \in [0, 100)$. Hence, $u = M$ for all t, and no singular intervals exist.

Part 2: Adding the new constraint as a differential equation $\dot{z} = u$ with initial condition $z(0) = 0$ and terminal condition $z(t_f) = K$, we have the following expression for the Hamiltonian

$$H = -x - 0.1\lambda_1 x + \lambda_1 u + \lambda_2 u$$

The costate equations become

$$\begin{aligned} \dot{\lambda}_1 &= 1 + 0.1\lambda_1 \\ \dot{\lambda}_2 &= 0 \end{aligned}$$

From the terminal conditions, $x(t_f)$ is free but $z(t_f)$ is fixed, so $\lambda_1(t_f) = 0$ and λ_2 is free. $\lambda_1(t)$ can now be determined as in Part 1, but λ_2 requires the solution of the TPBVP.

The optimal control is

$$u = \arg\min_u H = \begin{cases} M & \text{if } \lambda_1 + \lambda_2 < 0 \\ 0 & \text{if } \lambda_1 + \lambda_2 > 0 \end{cases}$$

Without solving the complete system, it is obvious that $\lambda_1 + \lambda_2$ is an increasing exponential (no singular intervals) and there are the following possibilities:

1. $(\lambda_1 + \lambda_2)|_0 < 0$ and $(\lambda_1 + \lambda_2)|_{100} \leq 0$: $u = M$ in $[0, 100]$. Apply the maximum input; the water supply is never depleted or it is depleted at precisely the final time ($K/M \leq 100$). Notice that in the former case, the water supply constraint is not achieved and, in a strict sense, the problem is infeasible.

2. $(\lambda_1 + \lambda_2)|_0 < 0$ and $(\lambda_1 + \lambda_2)|_{100} > 0$: $u = M$ in $[0, t_1]$ and $u = 0$ afterwards. The time t_1 is determined as the time where $\lambda_1(t_1) + \lambda_2(t_1) = 0$, which happens when the water source is depleted. So, the optimal policy is to apply maximum flowrate until the supply is depleted.

3. $(\lambda_1 + \lambda_2)|_0 > 0$. That means $u = 0$, and it can only happen if $K = 0$ (there is no water supply).

Part 3: Here we have the following expression for the Hamiltonian

$$H = -0.1\lambda_1 x + \lambda_1 u + \lambda_2 u$$

The costate equations become

$$\begin{aligned} \dot{\lambda}_1 &= 0.1\lambda_1 \\ \dot{\lambda}_2 &= 0 \end{aligned}$$

From the terminal conditions, $x(t_f)$ is free but $z(t_f)$ is fixed, so $\lambda_1(t_f) = \frac{\partial(-x)}{\partial x}|_{t_f} = -1$ and λ_2 is free. The optimal control is again

$$u = \arg\min_u H = \begin{cases} M & \text{if } \lambda_1 + \lambda_2 < 0 \\ 0 & \text{if } \lambda_1 + \lambda_2 > 0 \end{cases}$$

but, in this case, $\lambda_1(0) = -e^{-0.1t_f}$ and $\lambda_1(t) + \lambda_2 = -e^{0.1(t-t_f)} + \lambda_2$. This means that $(\lambda_1 + \lambda_2)$ is a decreasing function of time and the optimal control is to use our water supply at the end of the time interval.

As in Part 2, we can distinguish different cases depending on the sign of $(\lambda_1 + \lambda_2)$ at 0 and 100. The critical parameter here is the input switching time $t_1 = t_f - K/M$ and the optimal policy is $u = M$ if $t > t_1$ and $u = 0$ otherwise. (A similar switching time could have been defined for Part 2, but the description of the optimal policy was much easier there.)

2.4 Shape of a hanging chain

The optimal control problem can be written as

$$\min_{u,y,s} \qquad \int_{x_0}^{x_f} y\sqrt{1+u^2}dx$$

$$
\begin{aligned}
\text{s.t.} \qquad & \frac{dy}{dx} = u \\
& \frac{ds}{dx} = \sqrt{1+u^2} \\
& x_0 = -l \;, \quad x_f = l \\
& y(x_0) = 0 \;, \quad y(x_f) = 0 \\
& s(x_0) = 0 \;, \quad s(x_f) = 2L
\end{aligned}
$$

The Hamiltonian for this problem is

$$
H = y\sqrt{1+u^2} + \lambda_1 u + \lambda_2 \sqrt{1+u^2}
$$

from which the necessary conditions for the optimal control are

$$
\begin{aligned}
& u: \; \frac{\partial H}{\partial u} = 0 \Rightarrow \frac{u}{\sqrt{1+u^2}} = -\frac{\lambda_1}{y+\lambda_2} \\
& \frac{d\lambda_1}{dx} = \sqrt{1+u^2} \Rightarrow \lambda_1 = -s + c_1 \\
& \frac{d\lambda_2}{dx} = 0 \Rightarrow \lambda_2 = \text{const.}
\end{aligned}
$$

Anticipating symmetry about the mid-point $x = 0$, we conjecture $s(0) = L$, $u(0) = 0$, and $y(0) = y_{min}$. These also imply that $\lambda_1(0) = 0$ (since $\frac{\partial H}{\partial u} = 0$) and $c_1 = L$. Thus, we have

$$
\lambda_1 = L - s \quad ; \quad \frac{u}{\sqrt{1+u^2}} = \frac{s-L}{y+\lambda_2}
$$

Next, we eliminate x from the differential equations of y and s:

$$
\frac{dy}{ds} = \frac{s-L}{y+\lambda_2} \Rightarrow (y+\lambda_2)^2 = (s-L)^2 + c_2
$$

where the last equation is derived after a straightforward separation of variables. To compute the constant c_2 we notice that at the end-points $y = 0$ so $c_2 = \lambda_2^2 - L^2$. At the mid-point $s = L$ so $c_2 = (y_{min} + \lambda_2)^2 > 0$. Thus, $\lambda_2^2 > L^2$. At the mid-point the Hamiltonian (which is constant as a function of x) has the value $\lambda_2 + y$ and we can define

$$
H^2 = c_2 = \lambda_2^2 - L^2
$$

Moreover, from the second order conditions for the minimum, the minimizer of the Hamiltonian should satisfy $\partial^2 H/\partial u^2 \geq 0$ which, in turn, yields $\lambda_2 + y \geq 0$ and $H = +\sqrt{\lambda_2^2 - L^2}$.

Substituting back in the equation for the optimal control

$$
\frac{u}{\sqrt{1+u^2}} = \frac{\lambda_1}{y-\lambda_2} = \frac{s-L}{\sqrt{H^2+(s-L)^2}} = \frac{s/H - L/H}{\sqrt{1+(s/H-L/H)^2}}
$$

where the square root sign is taken to be consistent with $u(x_0) < 0$, $u(x_f) > 0$. Thus,

$$
u = \frac{s}{H} - \frac{L}{H}
$$

which yields a differential equation for u that depends on no other variables:

$$
\frac{du}{dx} = \frac{1}{H}\frac{ds}{dx} = \frac{1}{H}\sqrt{1+u^2} \Rightarrow \frac{du}{d(x/H)} = \sqrt{1+u^2}
$$

The solution of this differential equation is[1]

$$
u = \sinh(x/H) + c_3
$$

[1] Recall that $\cosh^2(x) - \sinh^2(x) = 1$, $d\sinh x/dx = \cosh x$, $d\cosh x/dx = \sinh x$.

And $c_3 = 0$ since $u(0) = 0$ at the mid-point. Furthermore, at the end-point x_f, $(s-L)/H = L/H = u(x_f) = \sinh(l/H)$. So, the constant H should satsify

$$\frac{L}{H} = \sinh\left(\frac{l}{H}\right)$$

Finally, integrating $\frac{dy}{dx} = u$ we get that $y(x) = H\cosh(x/H) + c_4$. Using the end-point condition $y(l) = y(-l) = 0$, it follows that $c_4 = -H\cosh(l/H)$. (Notice that cosh is an even function.) Thus,

$$y(x) = H\left(\cosh\left[\frac{x}{H}\right] - \cosh\left[\frac{l}{H}\right]\right)$$

It is interesting to observe that the maximum deflection of y, occuring at the mid-point, is $-y_m in = H(\cosh(l/H) - 1)$. Normalizing L to 1 and letting $l \to L$ we have that $l/H \to 0$ and, by a Taylor expansion,

$$\begin{aligned} \frac{1}{H} &\simeq \frac{l}{H} + \frac{l^3}{6H^3} \Rightarrow H \simeq l\sqrt{\frac{l}{6(1-l)}} \\ -y_{min} &\simeq H\frac{1}{2}\frac{l^2}{H^2} \simeq \sqrt{\frac{3}{2}}\sqrt{\frac{1-l}{l}} \end{aligned}$$

Clearly, we must have $y_{min}^2 + l^2 \leq 1$ so this approximation can be meaningful only for $(1+l)l > 3/2$ (i.e., $l \sim 0.85L$ or more). Notice that $y_{min} \to 0$ as $\sqrt{1-l}$. So for a running length difference of only 1% ($l = 0.99$) the maximum deflection is 12.3% ($y_{min} = 0.123$)!

The "catenary" shape of the hanging chain is very close to a quadratic for medium-large l. But when l becomes small the nature of cosh becomes dominant and y takes a U-shape.

The analytic computation of y relies on the solution of $\frac{L}{H} = \sinh\left(\frac{l}{H}\right)$ for H. This is easily achieved by re-writing the equation as $\sinh^{-1}(ha) = h$ where $h = l/H$ and $a = L/l$. In MATLAB, $\sinh^{-1}$ is the built-in function `asinh`. Now, in the vicinity of the solution, $|\nabla_h \sinh^{-1}(ha)| = |a/\nabla_{ha}\sinh(h)| = |1/\cosh(h)| < 1$. Since the map $h = \sinh^{-1}(ha)$ is a local contraction, the iteration $h_{k+1} = \sinh^{-1}(ah_k)$ converges to the solution, locally. (In fact, the iteration converges to the positive root of the equation for any positive initial condition. But, depending on a, the convergence can be very slow.)

2.5 Min surface area of soap film

a. The Hamiltonian in Bolza form and the augmented performance index are

$$H = r\sqrt{1+u^2} + \lambda u, \quad \Phi = \nu[r(\ell) - a]. \tag{1}$$

The optimality condition is

$$0 = \frac{ru}{\sqrt{1+u^2}} + \lambda. \tag{2}$$

Since H is not an explicit function of x, H = constant is an integral of the TPBVP. Use this integral in place of the adjoint equation.

Substituting (2) into (1) and solving for u gives

$$u = \frac{\sqrt{r^2 - H^2}}{H}. \tag{3}$$

Substituting (3) into $dr/dx = u$ gives

$$\frac{dx}{H} = \frac{dr}{\sqrt{r^2 - H^2}},. \tag{4}$$

Integrating (4) gives

$$r = H\cosh\frac{x}{H} \quad \Rightarrow u = H\sinh\frac{x}{H}. \tag{5}$$

Evaluating (5) at $x = \ell$ gives

$$\frac{a}{\ell} = \frac{H}{\ell} \cosh \frac{\ell}{H}, \tag{6}$$

which is a transcendental equation that determines H/ℓ given a/ℓ.
b. The solution of (6) can be considered as the intersection of two functions of ℓ/H, the straight line $y = (a/\ell)(\ell/H)$ and the curve $y = \cosh(\ell/H)$. There will be two solutions for $\ell/a < .663$, one solution for $\ell/a = .663$ when the line is just tangent to the curve, and no solutions for $\ell/a > .663$.
c. Substituting (5) into the expression for the minimum area gives

$$A = 2\pi \int_{-\ell}^{\ell} H \cosh^2 \frac{x}{H} dx = 2\pi H \left[\ell + \frac{H}{2} \sinh \frac{2\ell}{H} \right], \tag{7}$$

which is a complicated function of ℓ/a [determined from (6) given ℓ/H]. Using (6) in (7) gives

$$A_{min} = 2\pi a^2 [\tanh(\ell/H) + (\ell/H)\operatorname{sech}^2(\ell/H)]. \tag{8}$$

A_{min} is a monotonically increasing function of ℓ/a in the range of interest, but it is greater than $2\pi a^2 =$ area inside the two loops for $\ell/H > .639$ which corresponds to $\ell/a > .528$. Thus

$$A_{min} = \begin{cases} 2\pi a^2 [\tanh(\ell/H) + (\ell/H)\operatorname{sech}^2(\ell/H)], & 0 \le \ell/a \le .528, \\ 2\pi a^2, & \ell/a \ge .528. \end{cases} \tag{9}$$

3

3.1

For this problem, the Hamiltonian is $H = (2x_1^2 + x_2^2 + u^2)/2 + \lambda_1 x_2 + \lambda_2(-x_1 + (1 - x_1^2)x_2 + u)$. Then, the costate equations are

$$\begin{aligned} \dot{\lambda}_1 &= -2x_1 + \lambda_2 + 2x_1 x_2 \lambda_2 \\ \dot{\lambda}_2 &= -x_2 - \lambda_1 - \lambda_2(1 - x_1^2) \end{aligned}$$

Minimizing H with respect to u we get

$$u = -\lambda_2$$

and the boundary conditions are $x(0) = x_0$, $x(1) = x_f$.

3.2

Here, $H = (3x^2 + u^2) + \lambda(-x + u)$. Hence,

$$\begin{aligned} \dot{\lambda} &= \lambda - 3x \\ u &= -\lambda \end{aligned}$$

Substituting back into $\dot{x} = -x + u$ we have

$$\frac{d}{dt} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & -1 \\ -3 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ \lambda \end{pmatrix}$$

which should be solved with boundary conditions $x(0) = x_0$ and $x(1) = 0$.

The solution of the above problem is

$$\begin{pmatrix} x \\ \lambda \end{pmatrix} (t) = e^{At} \begin{pmatrix} x_0 \\ \lambda_0 \end{pmatrix}$$

and the matrix exponential can be computed with a variety of methods (diagonalization, Laplace, Cayley-Hamilton).

Let $\phi_{ij}(t)$ denote the entries of e^{At}. Then

$$\begin{aligned} 0 = x(1) &= \phi_{11}(1)x_0 + \phi_{12}(1)\lambda_0 \ \Rightarrow \lambda_0 = -\phi_{12}^{-1}(1)\phi_{11}(1)x_0 \\ &\Rightarrow \ u(t) = -\lambda(t) = -\phi_{21}(t)x_0 + \phi_{22}(t)\phi_{12}^{-1}(1)\phi_{11}(1)x_0 \end{aligned}$$

If desired, this optimal control can be expressed in feedback form by solving the state/costate equations for x_0 in terms of $x(t)$ and substituting in the expression for u Notice that this control is only valid in the interval $[0, 1]$. If applied beyond that point, it will cause the state to diverge from 0.

A variant of this problem is to include a terminal cost $kx^2(t_f)/2$ and leave $x(t_f)$ free. Then, the terminal condition becomes $\lambda(t_f) = kx(t_f)$. Again, we can solve for λ_0 in terms of x_0:

$$\lambda_0 = (\phi_{22}(1) - k\phi_{12}(1))^{-1}(k\phi_{11}(1) - \phi_{21}(1))x_0$$

which for large k approaches the previous expression. These results generalize in the multidimensional case where the required inverses have been shown to exist under the assumptions of the LQR problem.

3.3 LQR Stabilization

Part 1. The solution of this LQR problem is $u = Kx = -R^{-1}B^\top Px$ where $P > 0$ satisfies the Algebraic Riccati equation

$$A^\top P + PA - PBR^{-1}B^\top P + Q = 0$$

Consider the function $V = x^\top Px$. Notice that $V \geq 0$ for all x and $V = 0$ if and only if $x = 0$. Evaluating the derivative of this function along the trajectories of the system,

$$\dot{V} = \dot{x}^\top Px + x^\top P\dot{x} = x^\top(A^\top P + PA)x + 2x^\top PBu$$

Observe that $x^\top PB = -u^\top R$. Then, using the Riccati,

$$\begin{aligned} \dot{V} &= x^\top(A^\top P + PA)x + x^\top PBu + x^\top PBu \\ &= x^\top(A^\top P + PA)x - x^\top PBR^{-1}B^\top Px - u^\top Ru \\ &= -x^\top Qx - u^\top Ru \leq 0 \end{aligned}$$

Since V is non-increasing, $V(t) \leq V(0)$ implying that $\lambda_{min}(P)\|x(t)\|^2 \leq \lambda_{max}(P)\|x_0\|^2$ and, hence, $\|x(t)\| \leq k\|x_0\|$ for some constant k and for any initial condition. This implies that the state-transition matrix of $A+BK$ is bounded and the zero-equilibrium is stable. In turn, the eigenvalues of $A+BK$ have non-positive real parts and any eigenvalues on the jw-axis are simple.

While the stability proof is fairly straightforward, asymptotic stability requires the use of a more elaborate argument from the theory of Lyapunov functions. This is based on the so-called La-Salle's theorem which, in the above setting, states that the system trajectories converge to a limit set characterized by $\dot{V} = 0$. (Notice that V is non-increasing and lower bounded, hence it reaches a limit.) The limit set contains trajectories of the system that satisfy $x^\top Qx = 0$, $u^\top Ru = 0$ and $\dot{x} = Ax + Bu$. Since $R > 0$, we have that $u = 0$. Furthermore, with $Q = C^\top C$, the limit set contains state trajectories such that $\dot{x} = Ax$ and $Cx = 0$. But since (A, C) is completely observable, the only trajectory contained in this set is $x = 0$. Hence $x \to 0$, implying that the zero-equilibrium is asymptotically stable; in turn, the eigenvalues of $A + BK$ have negative real parts.

Part 2. Repeating the stability argument with $u = \rho Kx$ we have

$$\begin{aligned} \dot{V} &= x^\top(A^\top P + PA)x + 2\rho x^\top PBu \\ &= x^\top(A^\top P + PA)x - x^\top PBR^{-1}B^\top Px + x^\top PBR^{-1}B^\top Px - 2\rho u^\top Ru \\ &= -x^\top Qx - (2\rho - 1)u^\top Ru \end{aligned}$$

Thus, $\dot{V} \leq 0$ provided that $2\rho - 1 > 0$ or $\rho > 1/2$. This implies that stability is preserved if the control input is multiplied by any constant in the interval $(1/2, \infty)$. Hence the upward gain margin of the LQR is infinity and the downward gain margin is 1/2.

3.4 Exponentially weighted LQR

Consider the transformation of variables

$$\tilde{x} = xe^{\delta t} \quad ; \quad \tilde{u} = ue^{\delta t}$$

Then,

$$\dot{\tilde{x}} = \delta\tilde{x} + e^{\delta t}(Ax + Bu) = (A + \delta I)\tilde{x} + B\tilde{u}$$

For this system we would like to minimize $J = \int_0^\infty (\tilde{x}^\top Q\tilde{x} + \tilde{u}^\top R\tilde{u})dt$. Using the standard LQR result, the optimal control is $\tilde{u} = -R^{-1}B^\top P\tilde{x}$, where $P > 0$ is the solution of the Riccati

$$(A + \delta I)^\top P + P(A + \delta I) - PBR^{-1}B^\top P + Q = 0$$

Transforming the input back to the original coordinates,

$$u = -R^{-1}B^\top Px$$

In other words, the solution to the exponentially weighted LQR problem involves a Riccati for the shifted matrix $A + \delta I$.

Furthermore, since the LQR is a stabilizing compensator, the eigenvalues of $(A + \delta I) + BK$ have non-positive real parts, implying that the eigenvalues of $A + BK$ have real parts less than $-\delta$ (stability margin of δ) and the states converge to zero as $e^{-\delta t}$. Alternatively, since the LQR cost is finite, $e^{2\delta t}x^\top Qx$, $e^{2\delta t}u^\top Qu$ are integrable, implying that $x \to 0$ as $e^{-\delta t}$. (The complete argument requires the use of a uniform continuity condition and the observability of (A, C).)

It is worthwhile to point out that in this case the Lyapunov stability analysis results in $\dot{V} \leq -2\delta V$. This yields immediately that $V(t) \leq e^{-2\delta t}V(0)$ and, thus, $\|x(t)\| \leq ke^{-\delta t}\|x_0\|$ where k is a constant that depends on the condition number (ratio of maximum/minimum eigenvalues) of P.

Finally, notice that the derivation of the above properties hinges on the controllability/observability of the shifted system, i.e., $(A + \delta I, B, C)$. It is straightforward to verify that these properties hold if (A, B, C) is c.c. and c.o. However, the weaker conditions for the solution of the LQR problem ((A, B, C) stabilizable and detectable) are not enough to guarantee the solution of the exponentially weighted LQR problem. In this case we need to assume that any uncontrollable or unobservable states have modes with real parts less than $-\delta$.

4

4.1 Time-optimal control

Here we want to find the optimal control that

$$\begin{aligned}
\min_{u,x} \quad & \int_0^{t_f} 1dt \\
\text{s.t.} \quad & \dot{x}_1 = x_2 \\
& \dot{x}_2 = u \\
& x(0) = x_0 \text{ arbitrary but given}, \quad x(t_f) = [1, 1]^\top \\
& |u(t)| \leq 1
\end{aligned}$$

The Hamiltonian for this problem is

$$H = 1 + \lambda_1 x_2 + \lambda_2 u$$

from which the necessary conditions for the optimal control are

$$\begin{aligned}
& u: \min_u(H) \Rightarrow u = -sign(\lambda_2) \\
& \dot{\lambda}_1 = 0 \Rightarrow \lambda_1 = \text{const.} \\
& \dot{\lambda}_2 = -\lambda_1 \Rightarrow \lambda_2 = -\lambda_1 t + c
\end{aligned}$$

From the boundary conditions $H(t) = H(t_f) = 0$ and for a singular interval $\lambda_2 \equiv 0$, hence $\lambda_1 = 0$, $c = 0$, implying $H = 1$, a contradiction; hence, no singular intervals exist. (This is expected since the system is controllable.)

Integrating with $u = \pm 1$ we get

$$\begin{aligned} x_2 &= \pm t + c_1 \\ x_1 &= \pm\frac{1}{2}t^2 + c_1 t + c_2 \end{aligned}$$

which yields

$$\begin{aligned} x_1 = \frac{1}{2}x_2^2 + c_3 \quad \text{if} \quad u = +1 \\ x_1 = -\frac{1}{2}x_2^2 + c_4 \quad \text{if} \quad u = -1 \end{aligned}$$

The trajectories that pass through x_f without any switchings are characterized by the constants $c_3 = 1/2$ (for $u = +1$) and $c_4 = 3/2$ (for $u = -1$). Hence, the switching curve can be defined as

$$S(x) = \begin{array}{rl} x_1 = \frac{1}{2}x_2^2 + \frac{1}{2} & \text{if} \quad x_2 < 1 \\ x_1 = -\frac{1}{2}x_2^2 + \frac{3}{2} & \text{if} \quad x_2 > 1 \end{array}$$

Notice that $S(x)$ is not differentiable at x_f. The switching curve divides the state-space into two halves, say S_- and S_+, defined as follows:

$$S_- = \left\{ \begin{array}{rl} x_1 - \frac{1}{2}x_2^2 - \frac{1}{2} > 0, & x_2 < 1 \\ x_1 + \frac{1}{2}x_2^2 - \frac{3}{2} > 0, & x_2 > 1 \\ x_1 \geq 1 & x_2 = 1 \end{array} \right.$$

$$S_+ = \left\{ \begin{array}{rl} x_1 - \frac{1}{2}x_2^2 - \frac{1}{2} < 0, & x_2 < 1 \\ x_1 + \frac{1}{2}x_2^2 - \frac{3}{2} < 0, & x_2 > 1 \\ x_1 < 1 & x_2 = 1 \end{array} \right.$$

The optimal control can now be written as

$$u = \left\{ \begin{array}{ll} -1, & x \in S_- \\ +1, & x \in S_+ \end{array} \right.$$

The action at x_f was chosen arbitrarily since it is not an equilibrium.

4.2 Weighted time-energy-optimal control

The Hamiltonian is $H = \alpha + u^2 + \lambda_1 x_2 + \lambda_2 u$. Then, the costate equations are

$$\begin{aligned} \dot{\lambda}_1 &= 0 \ \Rightarrow \lambda_1 = c_1 \ \text{(constant)} \\ \dot{\lambda}_2 &= -\lambda_1 \ \Rightarrow \lambda_2 = -c_1 t + c_2 \end{aligned}$$

Minimizing H with respect to u we get

$$u = -\lambda_2/2 \ \Rightarrow u = (c_1 t - c_2)/2$$

Now, if t_f is fixed then the optimal control problem is simply the transfer of an initial state to a final state minimizing the energy of the input. (Its solution can also be expressed as a minimum norm problem and is found in terms of the controllability Gramian, by performing a projection.) Using calculus of variations, we need to solve the above Euler equations with boundary conditions $x(0) = x_0$, $x(t_f) = x_f$. That is,

$$\begin{aligned} \dot{x}_2 &= u \ \Rightarrow x_2(t) = x_{2,0} - c_2 t/2 + c_1 t^2/4 \\ \dot{x}_1 &= x_2 \ \Rightarrow x_1(t) = x_{1,0} + x_{2,0}t - c_2 t^2/4 + c_1 t^3/12 \end{aligned}$$

Evaluating these expressions at t_f and setting $x_1(t_f) = x_{1,f}$ and $x_2(t_f) = x_{2,f}$ we can solve the two linear equations for the unknowns c_1, c_2.

If t_f is free, then the additional boundary condition is

$$H(t_f) = 0 = \alpha + (c_1 t_f - c_2)^2/4 + c_1 x_{1,f} - (c_2 - c_1 t_f)^2/2$$

We now have 3 polynomial equations for the unknowns c_1, c_2, t_f. (Multiple solutions may exist, but the minimizer can be identified by, e.g., computing the cost for each case.)

5.2 Furnace temperature ramp-up problem [Tsakalis and Stoddard, 6th IEEE Intl. Conf. ETFA, Los Angeles, 1997]

Problem statement: $\min t_f$ such that,

$\dot{x} = Ax + Bu, \quad y = Cx$

$x(0) = 0, \quad x(t_f) = x_f$ (st-st transition)

$-0.2 \le u \le 0.8$ (input constraints)

$-0.5 \le y_i - y_j \le 0.5$ (zone matching)

1. Compute x_f: $\quad y(t_f) = \begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix} \triangleq y_f$

$\dot{x}(t_f) = 0 = Ax(t_f) + Bu(t_f)$

$y(t_f) = Cx(t_f)$

$\Rightarrow x_f = -A^{-1}Bu_f, \quad y_f = -CA^{-1}Bu_f$

$\Rightarrow u_f = -(CA^{-1}B)^{-1} y_f$ (ss-input)

$x_f = A^{-1}B(CA^{-1}B)^{-1} u_f$ (ss-state)

2. Constraints: $(i = 1,2,3)$

$$\dot{z}_i = g_i(x) = \begin{cases} \left(x^T \tilde{C}_{ij}^T \tilde{C}_{ij} x - 0.5^2\right)^2 & \text{if } (\cdot) > 0 \\ 0 & \text{if } (\cdot) \le 0 \end{cases}$$

$z_i(0) = 0$

$\tilde{C}_{ij} x = y_i - y_j = (C_i - C_j)\, x = \tilde{y}_i$

Note: $\dfrac{\partial g_i}{\partial x} = 2\left(x^T \tilde{C}_i^T \tilde{C}_i x - 0.25\right) \tilde{C}_i^T \hat{C}_i x$ if $(\cdot) > 0$

0 otherwise

For computation, $x^T \tilde{C}_i^T \tilde{C}_i x = \tilde{y}_i^2$

$\tilde{C}_i x = \tilde{y}_i$ ①

Feasibility problem:

Given t_f, compute:

$$\min J = \frac{1}{2} \left\| \begin{bmatrix} x(t_f) \\ z(t_f) \end{bmatrix} - \begin{bmatrix} x_f \\ 0 \end{bmatrix} \right\|_Q^2$$

$$= \min \frac{1}{2} \left\{ q \, \| x(t_f) - x_f \|^2 + \| z(t_f) \|^2 \right\}$$

weight

s.t. $\dot{x} = Ax + Bu$

$\dot{z} = g(x)$

$u \in [-0.2, 0.8]$

If the min is 0 then t_f is feasible and $t_{min} \le t_f$. Otherwise $t_{min} > t_f$.

Gradient algorithm:

$H = \lambda^T (Ax + Bu) + \sum \lambda_i g_i(x)$

$\dfrac{\partial H}{\partial x} = A^T \lambda + \sum \lambda_i \dfrac{\partial g_i}{\partial x}$

$\dfrac{\partial H}{\partial z} = 0$

$\therefore$

⓪ Init u

① Solve $\dot{x} = Ax + Bu$ in $[0, t_f]$. $\tilde{u} = \tilde{C}$

e.g. lsim $(A, B, \tilde{C}, \tilde{D}, u, t)$ $\quad \tilde{y} = \tilde{C}x$

② BC for λ: $\quad \lambda(t_f) = x(t_f) - x_f$

$\lambda_i = z_i(t_f)$ (constants

③ Solve costates backwards in time ②

eg. $v_i = \lambda_i 2(y_i^2 - 0.25)\cdot y_i$ (can be nicely vectorized)

$v(\tau) = v(t_f - \tau)$ (Reverse time)

$\lambda = \mathrm{lsim}(A^T, \tilde{C}^T, I, 0, v, \lambda(t_f))$

$\lambda(\tau) = \lambda(t_f - \tau)$ (Reverse time)

Optimal input : $u^* = \mathrm{sign}_{[-0.2,\, 0.8]}(-B^T\lambda)$

Descent direction: $u_{k+1} = u_k + \gamma\,[u^* - u_k]$

- Perform a line search w.r.t. γ s.t. $J_{k+1} < J_k$

Comments :

- Quick coding
- Reasonable solutions for $t_f \gg t_{min}$ (good accuracy without excessive iterations)
- Very slow convergence near t_{min} !!
- Incorporate conjugate directions optimization to improve speed of convergence.
- Variation of extremals version ?? (TPBVP)

Convex Optimization Approach

* Special case problem !!

1. Convert to a discrete time problem

   $[A,B,C,D]_{CT} \xrightarrow{C2D} [A,B,C,D]_{DT}$

   for sampling times up to 5 sec the DT system response is nearly identical to CT

2. Compute impulse response functions h_{ij}

   Define $H_{ij} = \mathrm{toeplitz}(h_{ij})$

   s.t. $y_i = H_{ij}\, u_j$

   Then define $y = \mathrm{vec}(y_i)$, $u = \mathrm{vec}(u_i)$ (concatenation) s.t. $y = Hu$.

3. Steady state dfn:

   Let H_L the last n rows of H_{ij} s.t.

   $H_L u$ = last n outputs

   For steady state, we want $H_L u = y_f = \begin{pmatrix} 100 \\ 100 \\ 100 \\ \vdots \end{pmatrix}$

   (Instead of $\dot{x} = 0$ or $x_{k+1} - x_k = 0$. Here, x_k is not available and it is not convenient to recreate).

4. Constraints: All can be expressed as hyperplane constraints $\alpha u \le \beta$

   α, β appropriate matrices

- Control input : $\begin{pmatrix} I \\ -I \end{pmatrix} u \le \begin{pmatrix} 0.8 \\ \vdots \\ 0.2 \\ \vdots \end{pmatrix}$
- Output matching $\left([I, -I, 0]\, H\right) u \le 0.5$

$$\underbrace{\left([-I, I, 0]\, H\right)}_{\alpha} u \le \underbrace{0.5}_{\beta}$$

etc.

5) Convex problem : a Projection

$$\min \| H_L u - y_* \|$$

$$= \min \| H_L (u - u_*) \|$$

$$= \min \| u - u_* \|_{H_L^T H_L}$$

s.t. $\alpha u \le \beta$

$u_* = H_L \backslash y_*$
Exists by controllability but may not satisfy all constraints

Soln: $u = P_{\substack{\alpha u \le \beta \\ H_L^T H_L}} [u_*]$: Oblique projection of u_* on the polytope $\alpha u \le \beta$

---

6). Solution Comments

- High accuracy sol'n. Feasible $t_f \simeq 6$ min
- Speed vs sampling time

$T_s =$	5 sec	3 sec
Interior point Alg.	40 sec	225 sec
Exterior point Alg	64 sec	352 sec

- Interior point algorithms tend to produce smoother projections. But high-variance controls may present a general problem.
- Alternative formulation :

H = step response matrix

$u \to \delta u$ = input increments

Adjust constr. matrix : $\begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{pmatrix} u \le \begin{pmatrix} 0.8 \\ \vdots \\ 0.8 \end{pmatrix}$

etc.

Then, $\min \|\delta u - \delta u_*\|^2_{H_L^T H_L} + \rho \|\delta u\|^2$

$\Leftrightarrow$ oblique projection matrix $\rho I + H_L^T H_L$

By increasing ρ, the variation of the control input is penalized, trading off some of the accuracy of the final steady-state.

```
% STEEPEST DESCENT PROCEDURE
tf = input('tf ');
format short e
ti = [0:3/60:tf]';  N = length(ti);
ri = N:-1:1;
uk = zeros(N,3)+0.8;
niter=0; dd=.5*.5;

load isoc

sys = ss(A,B,C,D);
Ct=[C-C([2,3,1],:)]; Dt = [D-D([2,3,1],:)];
syst = ss(A,B,Ct,Dt);
sysa = ss(A',Ct',B',D');
sysi = ss(zeros(3,3),eye(3,3),eye(3,3),zeros(3,3));
xf = (A\B)*((C*(A\B))\[100;100;100]);
zf = [0;0;0];

[dy,t,x] = lsim(syst,uk,ti);
dy2 = dy.^2 -dd; ig = find(dy2 > 0); dg = 0*dy2; dg(ig) = dy2(ig); g = dg.^2;
z = lsim(sysi,g,ti); Lf = 100*(x(N,:)' - xf); Lzf = z(N,:)';
J0 = [Lf;Lzf]'*[Lf;Lzf]/2
upd=1;

while upd == 1
    dg_in = (2*dg.*dy)*diag(Lzf);
    dg_in = dg_in(ri,:);
    [tem,t,L] = lsim(sysa,dg_in,ti,Lf);
    L = L(ri,:); us = -L*B;
    us = max(-1e10*L * B,-.2); us = min(us,0.8);

    J = inf; gamma = 10;
    while J0 < J
        if gamma > 1e-10; gamma = gamma/2; else gamma = 0; disp('forced exit'); end
        u = (uk + gamma * [us - uk]);
        [dy,t,x] = lsim(syst,u,ti);
        dy2 = dy.^2 -dd; ig = find(dy2 > 0); dg = 0*dy2; dg(ig) = dy2(ig); g = dg.^2;
        z = lsim(sysi,g,ti); Lf = 100*(x(N,:)' - xf); Lzf = z(N,:)';
        J = [Lf;Lzf]'*[Lf;Lzf]/2;
    end
    if J0-J < 1e-10;
        upd = 0;
    else
        niter = niter + 1; J0=J; uk = u;
        disp([niter,J])
    end
end
y=lsim(sys,u,t);
plot(t,y,t,u*100)
```

```
% OBLIQUE PROJECTION SOLUTION
tf = input('tf ');
format short e
TS = 3/60;  SSS=4;
ti = [0:TS:tf]';  N = length(ti);
ri = N:-1:1;

load isoc
sys=ss(A,B,C,D);
sysd=c2d(sys,TS);
[y,k]=impulse(sysd);
h1=y(:,:,1);h2=y(:,:,2);h3=y(:,:,3);
h1=h1(1:N,:);h2=h2(1:N,:);h3=h3(1:N,:);
H1 = [ toeplitz(h1(:,1),[h1(1,1),zeros(1,N-1)]);
       toeplitz(h1(:,2),[h1(1,2),zeros(1,N-1)]);
       toeplitz(h1(:,3),[h1(1,3),zeros(1,N-1)])
     ];
H2 = [ toeplitz(h2(:,1),[h2(1,1),zeros(1,N-1)]);
       toeplitz(h2(:,2),[h2(1,2),zeros(1,N-1)]);
       toeplitz(h2(:,3),[h2(1,3),zeros(1,N-1)])
     ];
H3 = [ toeplitz(h3(:,1),[h3(1,1),zeros(1,N-1)]);
       toeplitz(h3(:,2),[h3(1,2),zeros(1,N-1)]);
       toeplitz(h3(:,3),[h3(1,3),zeros(1,N-1)])
     ];
H=[H1 H2 H3];
a=[eye(3*N,3*N);-eye(3*N,3*N)]; b=[ones(3*N,1)*.8;ones(3*N,1)*.2];
c1=[eye(N,N),-eye(N,N),zeros(N,N)];
c2=[eye(N,N),zeros(N,N),-eye(N,N)];
c3=[zeros(N,N),eye(N,N),-eye(N,N)];
bd=ones(N,1)*.5;
a=[a;c1*H;-c1*H;c2*H;-c2*H;c3*H;-c3*H];
b=[b;bd;bd;bd;bd;bd;bd];

HSS=H([[N-SSS:N],[2*N-SSS:2*N],[3*N-SSS:3*N]],:);
yss=ones(3*(SSS+1),1)*100;uss=HSS\yss;
HTH=HSS'*HSS+0*eye(3*N,3*N); disp('starting projection');tic
u=orpr(uss,HTH,a,b,[],[],1,1e-6,1);toc
```

```
% OBLIQUE PROJECTION SOLUTION II
tf = input('tf ');
format short e
TS = 5/60;  SSS=4;
ti = [0:TS:tf]';  N = length(ti);
ri = N:-1:1;

load isoc
sid=ss(eye(3,3),eye(3,3),eye(3,3),eye(3,3),-1);
sys=ss(A,B,C,D);
sysd=c2d(sys,TS);
[y,k]=step(sysd);
h1=y(:,:,1);h2=y(:,:,2);h3=y(:,:,3);
h1=h1(1:N,:);h2=h2(1:N,:);h3=h3(1:N,:);
H1 = [ toeplitz(h1(:,1),[h1(1,1),zeros(1,N-1)]);
       toeplitz(h1(:,2),[h1(1,2),zeros(1,N-1)]);
       toeplitz(h1(:,3),[h1(1,3),zeros(1,N-1)])
     ];
H2 = [ toeplitz(h2(:,1),[h2(1,1),zeros(1,N-1)]);
       toeplitz(h2(:,2),[h2(1,2),zeros(1,N-1)]);
       toeplitz(h2(:,3),[h2(1,3),zeros(1,N-1)])
     ];
H3 = [ toeplitz(h3(:,1),[h3(1,1),zeros(1,N-1)]);
       toeplitz(h3(:,2),[h3(1,2),zeros(1,N-1)]);
       toeplitz(h3(:,3),[h3(1,3),zeros(1,N-1)])
     ];
H=[H1 H2 H3];
a=[toeplitz(ones(N,1),[1,zeros(1,N-1)]);-toeplitz(ones(N,1),[1,zeros(1,N-1)])];
b=[ones(N,1)*.8;ones(N,1)*.2];
a=[a 0*a 0*a;0*a a 0*a;0*a 0*a a];b=[b;b;b];
c1=[eye(N,N),-eye(N,N),zeros(N,N)];
c2=[eye(N,N),zeros(N,N),-eye(N,N)];c3=[zeros(N,N),eye(N,N),-eye(N,N)];
bd=ones(N,1)*.5;
a=[a;c1*H;-c1*H;c2*H;-c2*H;c3*H;-c3*H]; b=[b;bd;bd;bd;bd;bd;bd];

HSS=H([[N-SSS:N],[2*N-SSS:2*N],[3*N-SSS:3*N]],:);
yss=ones(3*(SSS+1),1)*100;uss=HSS\yss;
HTH=HSS'*HSS+1e-1*eye(3*N,3*N); disp('starting projection');tic
du=orpr(uss,HTH,a,b,[],[],1,1e-6,1);toc
u=lsim(sid,unvector(du,N,3));
plot(ti,unvector(H*du,N,3),ti,u*100)
```