EEE582 Homework Problems

HW1

1. Write a state-space realization of the linearized model for the cruise control system around speeds $v = 40$ (Section 0.3, http://tsakalis.faculty.asu.edu/notes/models.pdf). Use MATLAB commands to find its transfer function.

2. Write a state-space realization of the linearized model for the inverted pendulum around the unstable equilibrium (Section 0.5, http://tsakalis.faculty.asu.edu/notes/models.pdf). Use MATLAB commands to find its transfer function.

3. Write a state-space realization of the linearized model for the inverted pendulum on a cart system around the unstable equilibrium (Section 0.6, http://tsakalis.faculty.asu.edu/notes/models.pdf). Use MATLAB commands to find its transfer function.

4. Write a state-space realization of the linearized model for the notch filter (http://tsakalis.faculty.asu.edu/notes/notch.pdf). Use MATLAB commands to find its transfer function. Discretize the model using finite differences ($\dot{y} \simeq (y(t_{k+1}) - y(t_k))/(t_{k+1} - t_k)$) for different sampling times $T = t_{k+1} - t_k$. Select three different values of T and use MATLAB commands to compare the frequency response of the continuous-time model with the discrete one.

HW2

5. Find all solutions of

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{2} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}, \text{ where } \boldsymbol{x} \in \boldsymbol{R}^{\mathbf{4}}$$

Which is the minimum norm one?

6. Find all minimizers of the 2-norm of

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{2} \\ \mathbf{4} & \mathbf{3} \end{bmatrix} \boldsymbol{x} - \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}, \text{ where } \boldsymbol{x} \in \boldsymbol{R}^{\mathbf{2}}$$

7. Compute the Jordan Canonical Form and A^{10} and the matrix exponential $\boldsymbol{e^{At}}$ for

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} \end{bmatrix}$$

(Note: You do not need to compute the eigenvector matrix, just J. Also, you are free to use any approach to compute the matrix exponential.)

8. If A is symmetric, what is the relationship between its eigenvalues and singular values?

9. For $a, b \in \boldsymbol{R}^n$, show that $\det(I + ab^T) = 1 + a^T b$.

HW3

10. Compute the matrix exponential using Laplace transform, Cayley-Hamilton, Jordan form, for

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{-2} \end{bmatrix}$$

11. Find the step response of the system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{-2} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \boldsymbol{u}, \quad \boldsymbol{y} = [\mathbf{1} \;\; \mathbf{1}]\boldsymbol{x} + [\mathbf{0}]\boldsymbol{u}$$

by finding its transfer function, the Laplace transform of the output, and inverting it; and by using the general solution of the state equations with the matrix exponential.

12. Consider the system $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}, \;\; \boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du}.$ Write state-space realizations for the feedback system with $\boldsymbol{u} = \boldsymbol{Kx} + \boldsymbol{Gr}$ and for the inverse system $\boldsymbol{y} \mapsto \boldsymbol{u},$ specifying any conditions necessary for their existence.

13. Consider the SISO system $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}, \;\; \boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du}$ and define the "relative degree r" as the difference between the degrees of denominator and numerator. Show that when r > 0, $\boldsymbol{CA^{r-1}B} \neq \mathbf{0}, \boldsymbol{CA^{i}B} = \mathbf{0}, \boldsymbol{i} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \boldsymbol{r} - \mathbf{2}, \boldsymbol{D} = \mathbf{0}.$

14. Transform a time-invariant $[\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}]$ into $[\mathbf{0}, \widetilde{\boldsymbol{B}}(\boldsymbol{t}), \widetilde{\boldsymbol{C}}(\boldsymbol{t})]$ by a time varying equivalence transformation. When is it a Lyapunov transformation?

15. Verify that a the solution of $\dot{\boldsymbol{X}} = \boldsymbol{AX} + \boldsymbol{XB}, \;\; \boldsymbol{X}(\mathbf{0}) = \boldsymbol{C}$ is $\boldsymbol{X}(\boldsymbol{t}) = \boldsymbol{e^{At}Ce^{Bt}}$

16. Show that $\frac{\partial \boldsymbol{\Phi}(\boldsymbol{t_0}, \boldsymbol{t})}{\partial \boldsymbol{t}} = -\boldsymbol{\Phi}(\boldsymbol{t_0}, \boldsymbol{t})\boldsymbol{A}(\boldsymbol{t})$

17. Use Lyapunov theory to determine the stability of the following four systems and verify by computing their eigenvalues

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-2} & \mathbf{-2} \end{bmatrix} \boldsymbol{x}, \quad \dot{\boldsymbol{x}} = \begin{bmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{-2} & \mathbf{-2} \end{bmatrix} \boldsymbol{x}, \quad \boldsymbol{x_{k+1}} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-2} & \mathbf{-2} \end{bmatrix} \boldsymbol{x_k}, \quad \boldsymbol{x_{k+1}} = \begin{bmatrix} \mathbf{0} & \mathbf{0.1} \\ \mathbf{-1} & \mathbf{-1} \end{bmatrix} \boldsymbol{x_k}$$

18. Show that the eigenvalues of A have real parts less than $-\boldsymbol{\mu} < 0$ iff there exists a positive definite solution M to $\boldsymbol{A^T M} + \boldsymbol{MA} + \mathbf{2}\boldsymbol{\mu M} = -\boldsymbol{I}.$ Similarly, show that the eigenvalues of A have magnitude less than $\boldsymbol{\mu} < 1$ iff there exists a positive definite solution M to $\boldsymbol{\mu^2 M} - \boldsymbol{A^T MA} = \boldsymbol{\mu^2 I}.$

HW4

19. Determine whether the following state-space realizations are c.c. and c.o.:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} x + [0] u$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x + [0] u$$

20. Show that the state equation

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$

is controllable only if the pair (A_{22}, A_{21}) is controllable.

21. **When is the state-space realization with complex eigenvalues**

$$\dot{x} = \begin{bmatrix} a_1 & d_1 \\ -d_1 & a_1 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

completely controllable?

22. Check the controllability and observability of

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & t \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

23. Realize

$$G(s) = \frac{s+1}{(s+1)(s+2)}$$

1. **as a completely controllable system.**
2. **as a completely observable system.**

24. Show that controllability is invariant under state feedback, that is $\dot{x} = Ax + Bu$ is c.c. iff $\dot{x} = (A + BK)x + Bu$ is c.c.

HW5

25. Show that if the system $[A, B, C, D]$ realizes $H(s) = \frac{s+2}{s^2+2s+1}$ and $\det(sI - A) = s^2 + 2s + 1$, then $[A, B]$ must be controllable and $[A, C]$ must be observable.

26. Consider $H(s) = \frac{s^2+s-2}{s^4+3s^3+s^2-3s-2}$ and its realization in the controller canonical form with dim(A)=4. Is this realization observable? Give a minimal realization of H(s).

27. Given the system $P = [A, B, C, D]$, determine a realization of the feedback system $r \mapsto y: y = P[u], u = r - y$. Use two approaches:

1. Consider the closed-loop system with transfer function $(I + P)^{-1}P$, realize each term separately (use Problem 27) and then write the realization of the cascade combination of the two terms.
2. Directly from the realization of P by using the definition of the feedback interconnections (the dimension of the feedback is the same as the dimension of P).
   Comment on the minimality of the two realizations

28. Given the systems $P = [A, B, C, D]$, $K = [F, G, H, J]$ determine a state-space realization of the feedback system where $y = P[u], u = r - K[y]$.

29. Consider the system with transfer function

$$\boldsymbol{H(s) = \frac{1}{s^2 + 2s + 4}}$$

and its state-space realization in the controllable canonical form. Write three other zero-state equivalent realizations such that

- they have the same dimension
- none is topologically equivalent with the canonical form
- only two are topologically equivalent among them

30. Consider the system with transfer function matrix

$$H(s) = \begin{bmatrix} \frac{-3s^2 - 6s - 2}{(s+1)^2} & \frac{s^3 - 3s - 1}{(s-2)(s+1)^3} & \frac{-2}{(s+1)^2(s-2)} \\ \frac{s}{s+1} & \frac{3s^2}{(s+1)^3} & \frac{-6s}{(s+1)^2(s-2)} \end{bmatrix}$$

1. Determine a state-space realization by appending the realizations of each individual term.
2. Compute the observability, controllability and Hankel matrices and determine the order of a minimal representation of the system.
3. Compute a minimal realization of the system using the Kalman transformations. (Use MATLAB for the necessary computations but do not use “mineral()” directly.)

31. Compute the balanced realization for the system (Use MATLAB for the necessary computations but do not use "obalral()" directly.)

$$\dot{x} = Ax + Bu,\ \ y = Cx + Du;\ A = \begin{bmatrix} -1 & -3 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = [1\ 0], D = 0.$$

32. Consider the Discrete Time system $H(z) = \frac{z}{(z-0.2)(z-0.5)}$. Compute its impulse response analytically and then apply Theorem 7.M7 to find a state-space realization. (Use r = 10 for the Hankel matrix.)

HW6

33. Consider the system:

$$\dot{x} = Ax + Bu,\ \ y = Cx + Du;\ A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, C = [2\ \ 0\ \ 0], D = 0.$$

Design a state feedback controller $u = pr - Kx$ such that the closed loop eigenvalues are placed at $(-1 \pm j)$ and it tracks asymptotically any step reference input r.

34. Consider the discrete-time system:

$$x(k+1) = Ax(k) + Bu(k),\ \ y(k) = Cx(k);\ A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, C = [2\ \ 0\ \ 0]$$

Design a state feedback controller to place the closed-loop eigenvalues at (0,0,0). Show that for any initial state the closed-loop zero-state response becomes identically zero for $k \geq 3$. (Dead-beat response).

35. Consider the system:

$$\dot{x} = \begin{bmatrix} 2 & 1 & & \\ & 2 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} u$$

Is it possible to design a state feedback controller to place the closed loop eigenvalues at (-2, -2, -1, -1)?
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36. Consider the double integrator system:

$$\dot{x} = Ax + Bu,\ \ y = Cx + Du;\ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1\ \ 0], D = 0.$$

Design a state feedback controller $u = -Kx$ to place the closed loop eigenvalues at (-1, -1). Describe the properties of the closed-loop input disturbance sensitivity $d \mapsto (d - Kx) = [A - BK, B, -K, I]$. Repeat for the closed-loop eigenvalues (-10,-10).

37. Consider the double integrator system:

$$\dot{x} = Ax + Bu, \; y = Cx + Du; \; A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \;\; 0], D = 0.$$

Design an observer-based controller to stabilize the system and compute the transfer function of the corresponding controller K(s). Use (-1,-1) as closed loop poles for the state feedback (-1, -1) as the observer poles. Repeat for observer poles (-10,-10). Describe the properties of the closed-loop input disturbance sensitivity $\left(I - K(s)P(s)\right)^{-1}$ and compare it with the sensitivity of the state-feedback solution of Problem 36.

38. Consider the double integrator system of Problem (37):

$$\dot{x} = Ax + Bu, \; y = Cx + Du; \; A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \;\; 0], D = 0.$$

Design an LQG controller to stabilize the system and compute the transfer function of the corresponding controller K(s). Choose the control/observer weights such that the control loop has sensitivity bandwidth 1 and the observer loop has bandwidth 1. Repeat for an observer bandwidth of 10. Describe the properties of the closed-loop input disturbance sensitivity $\left(I - K(s)P(s)\right)^{-1}$ and output noise sensitivity $\left(I - P(s)K(s)\right)^{-1}P(s)K(s)$. Compare with the sensitivity of the state-feedback solution of Problem 36.

39. A well-known difficult problem from classical feedback control is the stabilization of a transfer function with interlaced poles and zeros in the right-half plane. In this homework, we will investigate the solution of this problem using state feedback/observer methods. Consider the system with transfer function $G(s) = \frac{s-2}{(s-1)(s-3)}$

1. Write a state-space realization for *G(s)*.

2. Design a stabilizing state feedback and an observer for your realization.

3. Compute the corresponding transfer function of the dynamic output controller *K(s)* (y -> u) and examine the root locus of *GK(s).*

For your designs consider two cases:

1. State feedback eigenvalues at $-1 \pm j$, Observer eigenvalues $-0.2, -10$

2. State feedback eigenvalues at $-1 \pm j$, Observer eigenvalues $-1 \pm j$