EEE 582 HW#5 SOLUTIONS

#25: The numerator and denominator of H are coprime, so a minimal realization has order 2 and $\det(sI-A) = \mathrm{den}(H)$. Any realization of order 2 of H must be minimal, therefore, c.c. and c.o.

#26: The zeros of H are $\{1,-2\}$ and the poles $\{1,-2,-1,-1\}$. Since there are common roots, a realization of order 4 cannot be minimal. Since the t.f. is realized in the controller canonical form, it must be not c.o. For an observable + controllable (minimal) realization we can apply the KCF algorithm or, simply, perform the algebraic cancellations and realize the resulting system:

$$H(s) = \frac{1}{s^2+2s+1} \Rightarrow [A,B,C,D] = \left[\begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \end{pmatrix}, 0\right]$$

#27:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du \Rightarrow u = -D^{-1}Cx + D^{-1}y$$

$$\Rightarrow \dot{x} = Ax + B(-D^{-1}Cx + D^{-1}y)$$

$$= (A - BD^{-1}C)x + BD^{-1}y$$

$$\Rightarrow H^{-1} = \left[(A-BD^{-1}C),\ BD^{-1},\ -D^{-1}C,\ D^{-1}\right]$$

#28: 1) $I+P = [A, B, C, D+I]$

$$(I+P)^{-1} = \left[\left(A - B(I+D)^{-1}C\right),\ B(I+D)^{-1},\ -(I+D)^{-1}C,\ (I+D)^{-1}\right]$$

(provided that $(I+D)^{-1}$ exists; otherwise the feedback system is not well-defined).

The cascade interconnection of two subsystems H_1 and H_2 has the realization

$$\dot{x} = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} x + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} y_1$$

$$y = \begin{bmatrix} 0 & C_2 \end{bmatrix} x + D_2 u_2$$

$$u_2 = y_1 = \begin{bmatrix} C_1 & 0 \end{bmatrix} x + D_1 u.$$

$$\Rightarrow \dot{x} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u$$

$$y = \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} x + D_2 D_1 u$$

Substituting,

$$(I+P)^{-1}P \leftrightarrow \left[\begin{pmatrix} A & 0 \\ B(I+D)^{-1}C & A - B(I+D)^{-1}C \end{pmatrix}, \begin{pmatrix} B \\ B(I+D)^{-1}D \end{pmatrix}, \left((I+D^{-1})C, -(I+D)^{-1}C\right), (I+D)^{-1}(D)\right]$$

2). $\mathring{x} = Ax + Bu$

$y = Cx + Du$

$u = r - y = r - Cx - Du = -(I+D)^{-1}Cx + (I+D)^{-1}r$

(I+D must be invertible)

$$\Rightarrow \mathring{x} = \left(A - B(I+D)^{-1}C\right)x + B(I+D)^{-1}r$$

$$y = \left(C - D(I+D)^{-1}C\right)x + D(I+D)^{-1}r$$

Notice that for both cases the system must be square (# inputs = # outputs).

While neither realization is a priori minimal, for the second realization we can observe that its order is the same as the order of P. The first realization has order twice as the order of P, <u>so it cannot be minimal</u>, and it may involve pole-zero cancellations in the RHP, if P has zeros in the RHP. On the other hand, if P is minimal, it can be shown that the second realization is minimal ($t.f = \frac{n}{n+d}$ which are coprime if n, d are coprime; the same is true for MIMO systems)

#29 : We write ss-realizations for the forward and feedback paths:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

$$\dot{z} = Fz + Gy, \quad v = Hz + Jy$$

with interconnection $u = r - v$.

Substituting, $u = r - Hz - J(Cx + Du)$

$$= (1+JD)^{-1}[-JC, -H]\begin{pmatrix} x \\ z \end{pmatrix} + (1+JD)^{-1} r$$

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \left(\begin{array}{c|c} A - B(1+JD)^{-1}JC & B(I+JD)^{-1}H \\ \hline GC - GD(1+JD)^{-1}JC & F - GD(I+JD)^{-1}H \end{array}\right)\begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B \\ GD(1+JD)^{-1} \end{pmatrix} r$$

$$y = \left(C - D(I+JD)^{-1}JC \;\middle|\; -D(1+JD)^{-1}H \right)\begin{pmatrix} x \\ z \end{pmatrix} + D(1+JD)^{-1} r$$

Again, $(1+JD)^{-1}$ should exist otherwise the feedback system is not well-defined.

#30 : 1) We define the systems in MATLAB:

H_{11} = ss(tf([-3,-6,-2],[1,2,1]))

⋮

H_{13} = ss(tf([-2],[1,2,1])) * ss(tf(1,[1,-2]))

⋮

(many choices and possibilities are possible).

Then, we construct the system by defining the system object:

$$H = [H_{11}, H_{12}, H_{13}; H_{21}, H_{22}, H_{23}]$$

This is a 16-th order system

2) We compute controllability and observability matrices with the Matlab commands "ctrb" and "obsv".
The rank of Q_c is 9 and the rank of Q_o is 8.
The Hankel matrix is $Q_o Q_c$ whose rank is 7. This is the order of a minimal realization of the system (notice that $rk(Q_o Q_c) \leq \min(rk(Q_o), rk(Q_c))$.)

3). Apply the algorithm in the notes. Check by comparing the system responses (bode and step, if possible, because the system is not stable) A better way to check system equality is to plot the singular values of the error system (sigma(H-Hr)) that should be very small (~ -200dB). The singular values, however, will not give an indication of which terms are different, in case of a mistake in the computation. Notice that, in general, one cannot simply compare the state space realizations - even with Matlab's minreal - since the realizations can be different by a similarity transformation.

#31. The controllability gramian is (gram(H, 'c'))

$$W_c = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.1667 \end{bmatrix}$$

and the observability gramian is

$$W_o = \begin{bmatrix} 0.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$

The first transformation now is (based on W_c)

$$T_1 = \begin{bmatrix} 0.707 & \\ & 0.408 \end{bmatrix}$$

For the transformed system the observability gramian is $\bar{W}_o = \begin{pmatrix} 0.25 & \\ & 0.25 \end{pmatrix}$

so the second transformation is

$$T_2 = \sqrt{\sqrt{\begin{pmatrix} 0.25 & \\ & 0.25 \end{pmatrix}}^{-1}} = \begin{pmatrix} 1.414 & \\ & 1.414 \end{pmatrix}$$

The balanced system is

$$\left[\begin{pmatrix} -1 & -1.732 \\ +1.732 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (1, 0), 0 \right]$$

with gramians $\begin{bmatrix} 0.5 & \\ & 0.5 \end{bmatrix}$

(Notice that Matlab's "balreal" produces a different answer that still has the same gramians).

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% EEE 582, Problem 32
% Example on state-space realization from Hankel matrix data

Hd=tf([1 0],conv([1 -.2],[1 -.5]),1)
ir=impulse(Hd);
M=20;  %  make sure there are enough samples
ir = ir(2:M);   % define H starting from h(2) = CB instead of h(1) = D

% % could try some quick example from noisy step response data too!
% y=step(Hd,0:100);y=y+(rand(size(y))-0.5)*.1; plot(y);pause
% ir=diff(y);

H=hankel(ir); semilogy(svd(H),'x');pause

T=H(1:10,1:10);  TT=H(1:10,2:11); [K,S,L]=svd(T);
semilogx(diag(S),'x');pause  % use r = 10 and estimate the order N

N=input('Give system order [2] '); if isempty(N);N=2;end
K=K(:,1:N); L=L(:,1:N);
O=K*sqrt(S(1:N,1:N));  C=sqrt(S(1:N,1:N))*(L');  % compute the O/C matrices
a=pinv(O)*TT*pinv(C); b=(C(:,1)); c=O(1,:); Hh=ss(a,b,c,0,1);  % define the system
bode(Hd,Hh);pause  % compare with the original
step(Hd,Hh);pause
tf(Hh)
```