HW Solutions: Linearization of the Inverted pendulum model

The nonlinear torque pendulum model has been derived in the class notes and is given as

$$\ddot{\theta} + \epsilon\dot{\theta} + a\sin\theta = bT_n$$

where T_n is the applied torque normalized in the interval [-1,1], and $\epsilon = c/mL^2$ (friction parameter), $a = g/L$, $b = T_{max}/mL^2$ (control gain).

The linearization of the model is performed around a nominal trajectory (solution). For our case, this trajectory is $T_n = 0$, $\theta = \pi$, $\dot{\theta} = 0$. By defining u, y as the perturbations around the nominal trajectory, i.e., $u = T_n - 0$, $y = \theta - \pi$, and expand the nonlinear terms to first order:

$$(\ddot{y} + 0) + \epsilon(\dot{y} + 0) + a\sin(y + \pi) = b(u + 0)$$
$$\ddot{y} + \epsilon\dot{y} + a\left(\sin\pi + \frac{d\sin x}{dx}(\pi)y + H.O.T.\right) = bu$$

Thus, we obtain the local linearized model

$$\ddot{y} + \epsilon\dot{y} - ay = bu$$

Examples of performance indices (J) to drive the pendulum to the vertical position with as little energy as possible are:

Exact, fixed final time $J = \int_{t_0}^{t_f} u^2$, s.t. $y(t_f) = \dot{y}(t_f) = 0$

Approximate, fixed final time $J = y(t_f)^2 + \lambda\dot{y}(t_f)^2 + \rho\int_{t_0}^{t_f} u^2$

Approximate, fixed final time $J = \int_{t_0}^{t_f} y^2 + \rho u^2$

Asymptotic $J = \int_{t_0}^{\infty} y^2 + \rho u^2$

The cart/inverted-pendulum model has also been derived in the class notes. The nonlinear equations in terms of the cart displacement y and the pendulum angle θ are

$$(m + M)\ddot{y} = F - c_c\dot{y} - mL\ddot{\theta}\cos\theta + mL\dot{\theta}^2\sin\theta \tag{1}$$

$$mL^2\ddot{\theta} = -c_p\dot{\theta} - mgL\sin\theta - mL\ddot{y}\cos\theta \tag{2}$$

where c_x are friction coefficients, m, M are the pendulum and cart masses and L is the pendulum length. We linearize around the solution $y = \dot{y} = 0$, $\theta = \pi$, $\dot{\theta} = 0$, $F = 0$. We define the state variables $X = [q; \dot{q}]$, where $q = [y - 0; \theta - \pi]$ and $\dot{q} = [\dot{y} - 0; \dot{\theta} - 0]$ and the input $u = F - 0$ and use the expansions:

$$\begin{aligned}
\cos\theta &= \cos(\pi + q_2) = \cos\pi - \sin\pi q_2 + H.O.T. \simeq -1 \\
\sin\theta &= \sin(\pi + q_2) = \sin\pi + \cos\pi q_2 + H.O.T. \simeq -q_2 \\
\ddot{\theta}\cos\theta &= (\ddot{q}_2)(\cos(\pi + q_2)) \simeq -\ddot{q}_2 \\
\dot{\theta}^2\sin\theta &= (\dot{q}_2)^2(\sin(\pi + q_2)) \simeq 0
\end{aligned}$$

We now get the linearized equations

$$\begin{aligned}
(m + M)\ddot{q}_1 &= u - c_c\dot{q}_1 - mL\ddot{q}_2(-1) \\
mL^2\ddot{q}_2 &= -c_p\dot{q}_2 - mgL(-q_2) - mL\ddot{q}_1(-1)
\end{aligned}$$

Collecting the second order derivatives,

$$\underbrace{\begin{pmatrix} m + M & -mL \\ -mL & mL^2 \end{pmatrix}}_{E}\ddot{q} = \underbrace{\begin{pmatrix} -c_c & 0 \\ 0 & c_p \end{pmatrix}}_{R}\dot{q} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & mgL \end{pmatrix}}_{G}q + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{H}u$$

Solving for $\ddot{q}$ we get

$$\ddot{q} = E^{-1}R\dot{q} + E^{-1}Gq + E^{-1}Hu$$

Finally, expressing the equations in terms of the state variables X we get the linearized state equations:

$$\dot{X} = \underbrace{\begin{pmatrix} 0 & I \\ E^{-1}G & E^{-1}R \end{pmatrix}}_{A} X + \underbrace{\begin{pmatrix} 0 \\ E^{-1}H \end{pmatrix}}_{B} u$$

Examples of performance indices (J) to drive the pendulum to the vertical position with as little energy as possible are:

Exact, fixed final time $J = \int_{t_0}^{t_f} u^2$, s.t. $X(t_f) = 0$

Approximate, fixed final time $J = X(t_f)^\top X(t_f) + \rho \int_{t_0}^{t_f} u^2$

Approximate, fixed final time $J = \int_{t_0}^{t_f} X^\top QX + \rho u^2$

Asymptotic $J = \int_{t_0}^{\infty} X^\top QX + \rho u^2$

where $Q > 0$ or $Q \geq 0$ and $(A, \sqrt{Q})$ is observable.