EEE 587 TEST 2 SOLUTIONS

PR#1 Let $\bar{x} = e^{\delta t} x$, $\bar{u} = e^{\delta t} u$

Then $\dot{\bar{x}} = \delta e^{\delta t} x + e^{\delta t}\dot{x} = (\delta I + A) e^{\delta t} x + e^{\delta t} B u$

$= (\delta I + A)\bar{x} + B\bar{u}$

The optimal control problem is $\min J = \int \bar{x}^T C^T C \bar{x} + \bar{u}^T R \bar{u}$

with solution $P: \bar{A}^T P + P\bar{A} + C^T C - PBR^{-1}B^T P = 0$

$\bar{u} = R^{-1} B^T P \bar{x}$

Then, $u = R^{-1}B^T P x$ and

$P: (A+\delta I)^T P + P(A+\delta I) + C^T C - PBR^{-1}B^T P = 0$

From LQR theory u stabilizes x, so

$\bar{A} - BR^{-1}B^T P = A + \delta I - BR^{-1}B^T P$ is stable

$\operatorname{eig} \bar{A} - BR^{-1}B^T P = \operatorname{eig}(A - BR^{-1}B^T P) + \delta \leq 0$

$\Rightarrow \operatorname{eig} A - BR^{-1}B^T P \leq -\delta \Rightarrow x(t) \leq C e^{-\delta t}$

The conditions are $(\bar{A}, B)$, stabilizable, $(\bar{A}, C)$ detectable

R positive definite.

Stronger conditions are (A,B) c.c., (A,C) c.o.

PR#2 1. The solution is $u = -R^{-1}B^T P x$ with

$$-\dot{P} = A^T P + PA + C^T C - PBR^{-1}B^T P, \quad P(t_f) = F$$

2. If $P(t_f)$ satisfies $A^T P + PA + C^T C - PBR^{-1}B^T P = 0$ then $\dot{P} = 0$ and this is true in $[t_0, t_f]$

3. Hence the control obtained by solving

$$\min J = \frac{1}{2} x^T F x \Big|_{t+T} + \int_t^{t+T} x^T C^T C x + u^T R u$$

is the same as the solution for the infinite interval

PR#3 Let $P_\rho = \rho P$. Then $u = -\rho K x = -B^T P_\rho x$ and P_ρ satisfies the Riccati:

$$A^T P + PA + C^T C - PBR^{-1}B^T P = 0$$

(here $C^T C = I$, $R = I$, for simplicity)

$$\Leftrightarrow \quad A^T P_\rho + P_\rho A + \rho C^T C - P_\rho B B^T P_\rho \cdot \frac{1}{\rho} = 0$$

$$\Leftrightarrow \quad A^T P_\rho + P_\rho A + \underbrace{\left[\rho I + \left(1-\frac{1}{\rho}\right) P_\rho B B^T P_\rho\right]}_{Q_\rho} - P_\rho B B^T P_\rho = 0$$

Since the "effective-Q" matrix $Q_\rho = \rho I + (1-\frac{1}{\rho}) P_\rho B B^T P_\rho$ is PD when $1-\frac{1}{\rho} > 0$, or $\rho > 1$, it follows that (A, Q_ρ) is observable (trivially, because rank $Q_\rho = n$) and hence the solution is stabilizing. (The result holds in general for $Q = C^T C$, (A, C) c.o. and $R > 0$ and for ρ in $[\frac{1}{2}, \infty)$ but this proof is "easy").

PR#4 The solution has $-\dot{P} = A^T P + PA + C^T C - PBR^{-1}B^T P$. Using the identity $\dot{(P^{-1})} = -P^{-1}\dot{P}P^{-1}$ we get

$$\dot{(P^{-1})} = (P^{-1})A^T + A(P^{-1}) + (P^{-1})C^T C(P^{-1}) - BR^{-1}B^T$$

$$\Leftrightarrow \quad -\dot{(P^{-1})} = (-A^T)^T(P^{-1}) + (P^{-1})(-A^T) + BR^{-1}B^T - (P^{-1})C^T C(P^{-1})$$

which is a Riccati with the substitutions:

$A \leftarrow -A^T$

$C^T C \leftarrow BR^{-1}B^T$

$BR^{-1}B^T \leftarrow C^T C$
