EEE 481 HW 1 SOLUTIONS

1.10 A typical heat transfer model is

$$m c_p \mathring{T} = -hA(T - T_0) + \tilde{e}$$

where m is mass of the heated object, c_p is the specific heat, h is the heat transfer coefficient. A is the area T_0 is the ambient temperature and $\tilde{e}$ is the net supplied power. These models are usually expressed in terms of the temperature difference $T - T_0 = \tilde{T}$, so

$$m c_p \mathring{\tilde{T}} = -hA\tilde{T} + \tilde{e}$$

$$\Rightarrow \quad \mathring{\tilde{T}} = -\frac{hA}{mc_p}\tilde{T} + \frac{1}{mc_p}\tilde{e}.$$

It is unclear whether the given model assumes $T_0 = 0$ or 25 and whether $T(0) = 0$ or 25. We will work with $C = T - T_0$ (meaning that unforced solutions converge to $T = T_0$) and $C(0) = 0$ in the first case, $C(0) = 25$ in the second.

a From the given transfer functions $\tilde{e}$ has two components, one is supplied power (heating) and the other is the door state (cooling). Each of these terms is converted to net power by a proportionality constant. After division with mc_p the final model is $(C(t) = \tilde{T}(t))$

$$\mathring{C} = -0.5C + 2e - 2.5d \iff C(s) = \frac{2}{s+0.5}e(s) - \frac{2.5}{s+0.5}d(s)$$

The time constant is $\tau = \frac{1}{0.5} = 2$ min.

b For $e(t) = 5u(t)$, $d(t) = 0$ we have

$$C(s) = \frac{2}{s+0.5}\left[\frac{5}{s}\right] = \frac{20}{s} - \frac{20}{s+0.5} \Rightarrow c(t) = 20u(t) - 20e^{-0.5t}u(t)$$

At steady state, $c_\infty = 20$

c With nonzero initial conditions,

$$\dot{c} = -0.5c + 2e - 2.5d \Leftrightarrow sC(s) - c(0) = -0.5C(s) + 2e(s) - 2.5d(s)$$

$$\Rightarrow C(s) = \frac{2}{s+0.5}e(s) - \frac{2.5}{s+0.5}d(s) + \frac{c(0)}{s+0.5}$$

For $c(0) = 25$, $c(t) = 20u(t) - 20e^{-0.5t}u(t) + 25e^{-0.5t}u(t)$

$$= \left(20 + 5e^{-0.5t}\right)u(t)$$

d With $d = u(t-2)$ and $c_*(t)$ denoting the sol'n of (b)

$$C(s) = C_*(s) + \frac{-2.5}{s+0.5}\left[\frac{e^{-2s}}{s}\right]$$

$$\Rightarrow c(t) = c_*(t) - \mathcal{L}^{-1}\left\{\frac{2.5}{s(s+0.5)}\right\}\Bigg|_{t=t-2} = c_*(t) - \left\{\begin{array}{l}\cancel{10}\,5\,u(t-2)\\ -\cancel{10}\,5\,e^{-0.5(t-2)}u(t-2)\end{array}\right\}$$

$$= 20\left(1 - e^{-0.5t}\right)u(t) - \cancel{10}\left(1 - e^{-0.5(t-2)}\right)u(t-2)$$

$$\therefore c_\infty = \cancel{10}.\ 15$$

2.7 $E(z) = \frac{z}{(z+1)^2}$ FVT: $\lim_{k \to \infty} e_k = \lim_{z \to 1} (z-1) E(z)$

if $\lim e_k$ exists.

$\therefore \lim_{z \to 1} (z-1) E(z) = 0$ but $e_k = -k(-1)^k$ which does not have a limit.

$E(z) = \frac{z}{(z-1)^2}$ $\lim_{z \to 1} (z-1) E(z) = \lim_{z \to 1} \frac{z}{z-1} = \infty$

$e(k) = k (1)^k$: diverges as well.

$E(z) = \frac{z}{(z-0.9)^2} \Rightarrow \frac{1}{0.9} k(0.9)^k \to 0 = \lim_{z \to 1} (z-1) E(z)$

$E(z) = \frac{z}{(z-1.1)^2} \Rightarrow \frac{1}{1.1} k (1.1)^k$ diverges but $\lim_{z \to 1} (z-1) E(z) = 0$

2.11 **a** $z^2 Y(z) - \frac{3}{4} z\, Y(z) + \frac{1}{8} Y(z) = e(z) = \frac{1}{z-1}$

$$\Rightarrow Y(z) = \frac{1}{\underbrace{z^2 - \frac{3}{4}z + \frac{1}{8}}_{(z-0.5)(z-0.25)}} \cdot \frac{1}{z-1} = \frac{2.667}{z-1} + \frac{-8}{z-0.5} + \frac{5.333}{z-0.25}$$

eg. using MATLAB's "residue"

$$\overset{\mathcal{Z}^{-1}}{\Rightarrow} y(k) = 2.667\, u(k-1) - 8\,(0.5)^{k-1} u(k-1) + 5.333\,(0.25)^{k-1} u(k-1)$$

Evaluating this expression,

$y(0) = 0$
$y(1) = 2.667 - 8 + 5.333 = 0$
$y(2) = 2.667 - 8/2 + 5.333/4 = 0$
$y(3) = 2.667 - 8/4 + 5.333/16 = 1$
$y(4) = 2.667 - 8/8 + 5.333/64 = 1.75$

b Solving the recursion with $y(0) = y(1) = 0$.

$y(2) = e(0) + 3/4\, y(1) - 1/8\, y(0) = 0$
$y(3) = e(1) + 3/4\, y(2) - 1/8\, y(1) = 1$
$y(4) = e(2) + 3/4\, y(3) - 1/8\, y(2) = 7/4 = 1.75$.

etc.

c With nonzero I.C. we use the z-transform pphy.

$$\mathcal{Z}\{y(k+1)\} = z\left[\mathcal{Z}\{y(k)\} - y(0)\right]$$

Then, $z^2 Y(z) - 3/4\, z Y(z) + 1/8\, Y(z) = z^2 y(0) + z\, y(1) - 3/4\, z\, y(0)$

$$\Rightarrow Y(z) = \frac{z\,(z - 11/4)}{(z-0.5)(z-0.25)} = z\left[\frac{10}{z-0.25} - \frac{9}{z-0.5}\right]$$

$$\Rightarrow y(k) = 10(0.25)^k u(k) - 9(0.5)^k u(k)$$

Evaluating,

$$y(0) = 1$$
$$y(1) = -2$$
$$y(2) = -1.625$$
$$y(3) = -0.9688$$

The same result is obtained from the recursion, starting with $y(0) = 1$, $y(1) = -2$. An alternative to these computations is to use the state-space version of the ODE:

$$x_k = \begin{pmatrix} y_k \\ y_{k+1} \end{pmatrix} \Rightarrow x_{k+1} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1/8 & 3/4 \end{pmatrix}}_{A} x_k + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e_k$$

$$y_k = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{C} x_k$$

With $x_0 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, we can compute

$$y(0) = C x_0$$
$$y(1) = C A x_0$$
$$y(2) = C A^2 x_0 = -1.625$$
$$y(3) = C A^3 x_0 = -0.9688$$

etc.

2.15

$y(t) = \int^t x(\tau)\,d\tau$

Left side

a) $y_{k+1} = y_k + x_k T$

$zY - Y = TX$

b) $\Rightarrow Y(z) = \frac{T}{z-1} X(z)$

e) $y(k) = \sum_{i=-\infty}^{k-1} x_i$

Right side

c) $y_{k+1} = y_k + x_{k+1} T$

$zY - Y = zXT$

d) $\Rightarrow Y(z) = \frac{Tz}{z-1} X(z).$

f) $y(k) = \sum_{i=-\infty}^{k} x_i$

2.16 $\quad y_{k+1} = y_k + \frac{x_k + x_{k+1}}{2} . T$

Taking z-transform

$zY - Y = \frac{X + zX}{2} T \Rightarrow$

$\Rightarrow (z-1)\,Y(z) = \frac{T}{2}(z+1)\,X(z)$

$\Rightarrow \frac{Y(z)}{X(z)} = \frac{T}{2}\,\frac{z+1}{z-1}$ (Tustin equivalent of an integrator)

2.26 a) $A = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $C = (-2, 1)$, $D = 0$

$$\frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D = (-2,1)\begin{bmatrix} z & -1 \\ 0 & z-3 \end{bmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{-z+4}{z(z-3)}$$

NOTE: Matlab version:
A = ..., B = ... etc.
G = ss(A,B,C,D)
tf(G)

b) $T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ (sum + difference of states) $\tilde{x} = Tx$

$$T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\tilde{x}_{k+1} = T x_{k+1} = TAT^{-1}\tilde{x}_k + TBu_k$$

$$y_k = CT^{-1}\tilde{x}_k + Du_k$$

$$\tilde{A} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \tilde{C} = \frac{1}{2}[-3, -1], \quad \tilde{D} = 0$$

c) $\frac{Y(z)}{U(z)} = \frac{-z+4}{z(z-3)}$ (invariant under similarity transformations)

d) Eigenvalues of A, $\tilde{A}$ are 0, 3

Det. $A = 0 = \det \tilde{A}$

trace $A = 3 = \text{trace}\,\tilde{A}$.

Transfer function invariance.

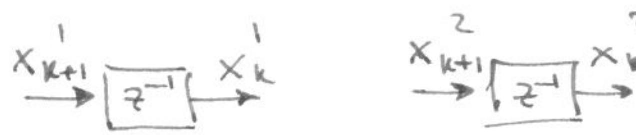
2.36 a) num = [3 4], den = [1 5 6]

[a, b, c, d] = tf2ss(num, den)

$$\rightarrow \quad a = \begin{bmatrix} -5 & -6 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad c = (3 \;\; 4), \quad d = 0$$

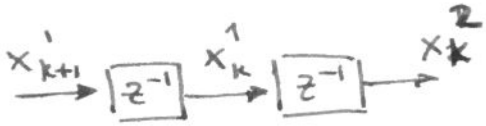
Alt. Gs = ss(G), a = Gs.a, b = Gs.b ... this produces a small difference in the realization $a = \begin{bmatrix} -5 & -6 \\ 2 & 0 \end{bmatrix}$ etc.

b) For the simulation diagram we start with the delays:

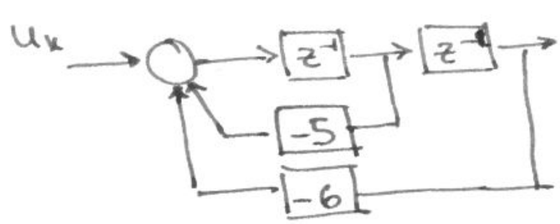


(superscripts denote element in vector)

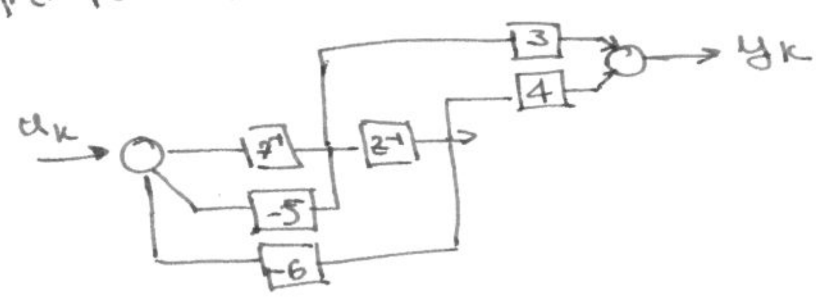
But $x^2_{k+1} = x^1_k$, so



Bring in the input: $x^1_{k+1} = -5x^1_k - 6x^2_k + u_k$



And form the output $y_k = 3x^1_k + 4x^2_k$



c) This simulation diagram is the same as Fig. 2.9 (controllable canonical form)

3.17 $G(s) = \frac{1-e^{-Ts}}{s}$, $\angle G = -\frac{\pi\omega}{\omega_s} + \Theta = -\frac{\omega T}{2} + \Theta$

$\Theta = \{ 0$ if $\sin \frac{T\omega}{2} > 0$, π otherwise

we expect (Nyquist) $\omega_s > \omega$, $\omega \in [0, 10\,Hz]$ so $\Theta = 0$

(ZOH is roughly half-sample delay)

The maximum lag appears at $\omega = 10\,Hz = 62.8\,rad/s$

and it is $\angle G = -5T \times 6.28 = -31.4T\ rad = 1800T°$

a). Lag $= 1800T < 10° \Rightarrow T < 1/180\ s$

b) Lag $= 1800T < 5° \Rightarrow T < 1/360\ s$

c) Lag $= 1800T < 20° \Rightarrow T < 1/90\ s$

Nyquist rate (10 Hz) $= 0.05\ s$ $\therefore$ $10°$ Lag is sampling about 10X Nyquist

3.24 a) 0 - 10V, 4 bit A/D

$V_{max} = V_{fs}\left(2^{-1}+2^{-2}+2^{-3}+2^{-4}\right) = 9.375\ V$

(resolution 0.0625×10 ; losing 1 bit from max)

b) $2^{-4} \times 10 = 0.625\ V$ min. possible Voltage

c) $V = 10\left(\alpha 2^{-1} + \beta 2^{-2} + \gamma 2^{-3}\right)$; $\alpha, \beta, \gamma \in [0,1]$

$= 0, 1.25, 2.5, 3.75, 5, 6.25, 7.5, 8.75$

d) $(2^{-n})\, 10 < 0.005 \Rightarrow -n \log 2 < \log 5\mathrm{E}{-4}$

$\Rightarrow n > + \dfrac{\log 2\mathrm{E}3}{\log 2} = 10.966$

$\Rightarrow n = 11 \quad \left(2^{11} \simeq 1\mathrm{E}{-3} \cdot \frac{1}{2}\right)$

3.25 a) Since A/D is a truncation (rather than rounding) max error is 1 bit occurring when the voltage is slightly **above** the next quantization level. So,

max error $= (2^{-8})\, 10 = 0.039\,V.$

b) 04_{HEX} corresponds to a voltage $(2^{-6})\, 10\ V = 0.156\ V.$

According to Fig. 3.22 the ADC is rounding up so the voltages producing 04_{HEX} are $(0.156 - 0.039,\ 0.156]\ V$

For a rounding down operation the interval would be

$[0.156,\ 0.156 + 0.039)\ V.$