EEE 582, TEST 2 **NAME:_____SOLUTIONS____**

Spring 2014, 60min, 3 Problems, Equal Credit, Closed-book, Closed-notes, calculator and 1 sheet of formulae allowed

Problem 1. Find a Lyapunov function $V(x) = x^T P x$ to show that the system $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix} x$ is asymptotically stable.

We solve the Lyapunov equation $A^T P + PA = -I$.
We find:

$$P = \frac{1}{16}\begin{bmatrix} 22 & 4 \\ 4 & 3 \end{bmatrix}$$

Since 22 > 0 and 22 x 3 – 4 x 4 = 50 > 0, by the Hurwitz test, P > 0. Hence V is a Lyapunov function for the given system.

Problem 2. Find the state transition matrix e^{At} for $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix} x$ (Use your favorite method).

For this problem the most efficient approaches are through Laplace transform and the Cayley-Hamilton theorem. For the latter, we have

$$e^{At} = b_0(t)I + b_1(t)A$$

And the coefficients are found by solving

$$e^{\lambda_1 t} = b_0(t) + b_1(t)\lambda_1$$
$$e^{\lambda_2 t} = b_0(t) + b_1(t)\lambda_2$$

Where, $\lambda_1 = -2 + \sqrt{2}$, $\lambda_2 = -2 - \sqrt{2}$. Thus,

$$b_1(t) = \frac{(e^{\lambda_1 t} - e^{\lambda_2 t})}{\lambda_1 - \lambda_2}, \quad b_0(t) = (e^{\lambda_1 t}) - \frac{\lambda_1}{\lambda_1 - \lambda_2}(e^{\lambda_1 t} - e^{\lambda_2 t})$$

For the Laplace transform approach,

$$e^{At} = L^{-1}\{(sI - A)^{-1}\} = L^{-1}\left\{\frac{1}{s^2 + 4s + 2}\begin{bmatrix} s+4 & 1 \\ -2 & s \end{bmatrix}\right\}$$

Next, we find the terms as linear combinations of $L^{-1}\left\{\frac{1}{s^2+4s+2}\right\}$ and its derivative.

$$L^{-1}\left\{\frac{1}{s^2 + 4s + 2}\right\} = L^{-1}\left\{\frac{(\lambda_1 - \lambda_2)^{-1}}{s - \lambda_1}\right\} + L^{-1}\left\{\frac{-(\lambda_1 - \lambda_2)^{-1}}{s - \lambda_2}\right\} = (\lambda_1 - \lambda_2)^{-1} e^{\lambda_1 t} + (\lambda_2 - \lambda_1)^{-1} e^{\lambda_2 t}$$

$$L^{-1}\left\{\frac{s}{s^2 + 4s + 2}\right\} = \frac{\lambda_1}{(\lambda_1 - \lambda_2)} e^{\lambda_1 t} + \frac{\lambda_2}{(\lambda_2 - \lambda_1)} e^{\lambda_2 t}$$

Performing the computations

$$e^{At} = \begin{bmatrix} \left(\frac{1}{\sqrt{2}} + \frac{1}{2} - \sqrt{2}\right) e^{(-2-\sqrt{2})t} + \left(-\frac{1}{\sqrt{2}} + \frac{1}{2} + \sqrt{2}\right) e^{(-2+\sqrt{2})t} & \left(-\frac{1}{2\sqrt{2}}\right) e^{(-2-\sqrt{2})t} + \left(\frac{1}{2\sqrt{2}}\right) e^{(-2+\sqrt{2})t} \\ \left(\frac{1}{\sqrt{2}}\right) e^{(-2-\sqrt{2})t} + \left(-\frac{1}{\sqrt{2}}\right) e^{(-2+\sqrt{2})t} & \left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right) e^{(-2-\sqrt{2})t} + \left(-\frac{1}{\sqrt{2}} + \frac{1}{2}\right) e^{(-2+\sqrt{2})t} \end{bmatrix}$$

Problem 3. Find the step response of the system [A,B,C,D]

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

when the initial condition is

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The key issue here is to compute the response due to the initial conditions (ZIR). The complete solution has the form

$$y(t) = Ce^{At}x_0 + \int Ce^{A(t-\tau)}Bu(\tau)d\tau = ZIR + ZSR$$

The ZIR can be easily computed using the previous expression for the matrix exponential and, for the given C and x0, it is the sum of the first row elements

$$ZIR = \left(\frac{1}{2\sqrt{2}} + \frac{1}{2} - \sqrt{2}\right)e^{(-2-\sqrt{2})t} + \left(-\frac{1}{2\sqrt{2}} + \frac{1}{2} + \sqrt{2}\right)e^{(-2+\sqrt{2})t}$$

The ZSR, on the other hand, could be easier to compute through the transfer function

$$ZSR = L^{-1}\left\{[C(sI-A)^{-1}B + D]\frac{1}{s}\right\} = L^{-1}\left\{\frac{1}{s(s-\lambda_1)(s-\lambda_2)}\right\}$$

Computing the partial fraction expansion,

$$ZSR = L^{-1}\left\{\frac{\frac{1}{2}}{s} + \frac{\frac{1}{2\sqrt{2}(-2+\sqrt{2})}}{(s-\lambda_1)} + \frac{\frac{1}{2\sqrt{2}(2+\sqrt{2})}}{(s-\lambda_2)}\right\} = \frac{1}{2} + \frac{1}{-4\sqrt{2}+4}e^{(-2+\sqrt{2})t} + \frac{1}{4\sqrt{2}+4}e^{(-2-\sqrt{2})t}$$

The complete step response is

$$y(t) = \frac{1}{2} + \left(\frac{1}{2\sqrt{2}} + \frac{1}{2} - \sqrt{2} + \frac{1}{4\sqrt{2}+4}\right)e^{(-2-\sqrt{2})t} + \left(-\frac{1}{2\sqrt{2}} + \frac{1}{2} + \sqrt{2} + \frac{1}{-4\sqrt{2}+4}\right)e^{(-2+\sqrt{2})t}, \qquad for\ t \geq 0$$