

Problem 1. Show that $\dot{x} = Ax; \quad y = Cx$ being completely observable is equivalent to $\dot{x} = Ax + Ly; \quad y = Cx$ being completely observable.

Let (A, C) be c.o. Then, for any set of complex conjugate eigenvalues, there exists L_o such that the eigenvalues of $A + L_oC$ are at the desired locations. Choose $L_1 = L_o - L$. Then, the eigenvalues of $(A + LC) + L_1C = A + L_oC$ are at the desired locations. Hence, $(A+LC, C)$ is c.o. Similarly for the converse.

Problem 2. Determine whether the following state-space realization is c.c. and c.o.:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u, \quad y = [0 \quad 2 \quad 2]x + [0]u$$

We compute the controllability and observability matrices:

$Q_c = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -3 \\ 0 & -3 & 8 \end{bmatrix}$, which has obviously independent columns, or simply its det is -8 (not zero). Hence, the system is c.c.

$Q_o = \begin{bmatrix} 0 & 2 & 2 \\ -2 & -6 & -4 \\ 4 & 10 & 6 \end{bmatrix}$, for which the det is 0 (or row1 = -2 row2-row3). Hence the system is not completely observable.

Problem 3. Find conditions for the pair $[A,B]$ to be completely controllable, when

$$A = \begin{bmatrix} a_1 & 0 & 1 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

We use the test that $[\lambda I - A | B]$ should be full rank for all λ and, in particular, eigenvalues of A. Here, the eigenvalues are simply the diagonal elements, but we have to distinguish cases for multiple eigenvalues.

1. Distinct eigenvalues. Then, for $\lambda = a_1$, we obtain the condition $b_3 - b_1(a_3 - a_1) \neq 0$. For $\lambda = a_2$, we obtain the condition $b_2 \neq 0$. For $\lambda = a_3$, we obtain the condition $b_3 \neq 0$.
2. All eigenvalues identical. Then there is no possible value of B to result in 3 independent columns. In this case, $[A, B]$ is never c.c.
3. $a_1 = a_2$, as in Case 2, $[A, B]$ is not c.c. for any values of A, B.
4. $a_3 = a_2$, as in Case 2, $\lambda = a_2 = a_3$, two columns of $[\lambda I - A | B]$ become zero (after an elementary column operation), so it cannot have rank 3, for any values of A, B. Hence $[A, B]$ is never c.c.

Summarizing, the only possibility for c.c. is that the eigenvalues are distinct and the conditions of Case 1 hold $b_3 - b_1(a_3 - a_1) \neq 0, b_2 \neq 0, b_3 \neq 0$. (These are not identical to the diagonal case conditions.)