Bernoulli's Principle - Advanced Discussion

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Introduction

This advanced discussion uses algebra and calculus to explain Bernoulli's Principle for the Bernoulli's Principle Animation. An introductory discussion is available for visitors who are not comfortable with algebra and calculus.

Ram Pressure and Dynamic Pressure

The demo and the other sections in this web page refer to the *pressure*. More specifically, they are referring to *static pressure*, which is the pressure felt by an object or person suspended in the fluid and moving with it. This pressure is static because the suspended object or person is not moving relative to the fluid. In this section only we will discuss two other types of pressure: *ram pressure* and *dynamic pressure*.

Static pressure should not be confused with ram pressure, which is the pressure felt by an object because it is moving relative to the fluid. Basically, the fluid is ramming into the moving object, or vice versa.

The ram pressure **increases** when the velocity **increases**. This explains the stronger force felt by your hand when it is held in a fast moving current. In the faster current, your hand is deflecting more flowing fluid from its original path.

As you wade across a rushing stream, the force against your legs is from the ram pressure, and it is directed downstream.

The static pressure **decreases** when the velocity **increases**, as explained in the Background Theory section. This explains why the water stream coming out of a firefighter's hose gets narrower a short distance past the nozzle - the stronger atmospheric pressure overwhelms the weaker static pressure in the quickly flowing water and compresses the water stream.

At the bottom of a swimming pool, the force on your body is from the static pressure of the water, and it is directed inwards.

Dynamic pressure corresponds to the movement of the fluid through the pipe, and is simply another name for the velocity terms in Bernoulli's Equation below. This pressure is dynamic because the fluid is moving relative to the pipe. The static pressure and the dynamic pressure are added to get the total pressure. In the demo graph under the Bernoulli's Principle Animation, the red shading corresponds to the static pressure, and the blue shading corresponds to the dynamic pressure. The sum of the static and dynamic pressures remains constant along the pipe.

In the initial pipe configuration, the static pressure on the right side is higher than the static pressure on the left. This does not mean the fluid is flowing from a lower pressure towards a higher pressure, since the total pressure is the same at every point.

Background Theory (Algebra and Calculus)

To understand how and why Bernoulli's Principle works, we need to look at Bernoulli's Equation. This equation can be summarized as follows:

> **Bernoulli's Equation** tells us how much the pressure within a moving fluid increases or decreases as the speed of the fluid changes. Here is Bernoulli's equation:

$$P_a + \frac{1}{2}\rho v_a^2 + \rho g h_a = P_b + \frac{1}{2}\rho v_b^2 + \rho g h_b$$

where

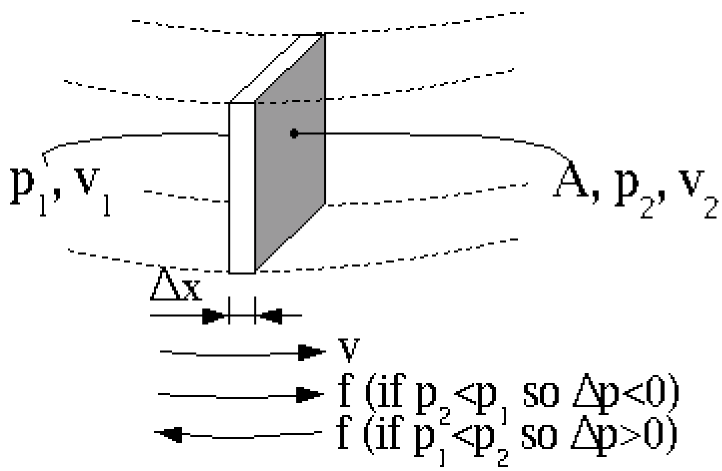
a is the first point along the pipe
b is the second point along the pipe
P is static pressure in newtons per meter squared
ρ is density in kilograms per meter cubed
v is velocity in meters per second
g is gravitational acceleration in meters per second squared
h is height in meters

There are two ways to derive Bernoulli's Equation: using Newton's Second Law or using the conservation of energy. The first approach implies that a change in pressure creates a change in velocity. The second approach implies that a change in velocity creates a change in pressure. So what is the cause, and what is the effect? The answer depends on the situation, and not the derivation. At least theoretically, any quantity can be changed in the physical sciences, making that quantity's change the "cause", and resulting changes in other quantities the "effect". For more information, read about cause and effect laws in the physical sciences.

Further information about the steps followed to compute the curves may be found at computing the curves.

Derivation Using Newton's Second Law (Calculus)

Many textbooks and web pages derive Bernoulli's Equation by starting with the conservation of energy. According to this approach, the pressure drops in reaction to an increase in fluid velocity. However, the following derivation starts with Newton's Second Law, which is a familiar starting point for many physics problems. As a bonus, this approach offers a more intuitive interaction between pressure and acceleration.



To derive Bernoulli's Equation, we first apply Newton's Second Law to the fluid in a segment of the pipe. During a particular time interval, the fluid travels the length of the segment.

$$f = ma$$

where
f - force (newtons)
m - mass of fluid (kilograms)
a - acceleration (meters / second^2)

The force arises because of the difference in pressure at either end of the segment, and the acceleration is related to the change in velocity. Since the force is in the direction of decreasing pressure, a minus sign is required.

$$-A\ \Delta P = m\frac{\Delta v}{\Delta t}$$

where
A - area (meter^2)
ΔP - static pressure difference (newtons per meter^2)
Δv - fluid velocity difference (meters per second)
Δt - time interval (seconds)

Replacing the mass and area by the density and the segment length gives:

$$\Delta P = -\rho\ \Delta x\ \frac{\Delta v}{\Delta t}$$

where
ρ - density of fluid (kilograms per meter^3)
Δx - segment length (meters)

Since the fluid travels the length of the segment during the time interval, we introduce the velocity to arrive at an equation that is ready for integration:

$$\int_{P_a}^{P_b} \Delta P = -\rho \int_{v_a}^{v_b} v\ \Delta v$$

where
v - velocity of fluid (meters / second)

Integrating, we arrive at Bernoulli's Equation:

$$P_b - P_a = \frac{1}{2}\rho\left(v_a^2 - v_b^2\right)$$

Since the pressure difference between two points creates the force that accelerates the fluid, we see that increasing the pressure difference increases the fluid velocity difference.

If Bernoulli's Equation is applied to a small volume of fluid, then each term in that equation can be multiplied by the volume V. This transforms Bernoulli's Equation into an "energy equation", where each term corresponds to some type of energy (potential, kinetic or pressure). Note that this derivation using Newton's Second Law does not discourage *applying* Bernoulli's Equation in its energy equation form, but this derivation offers a more intuitive way of *constructing* Bernoulli's Equation.

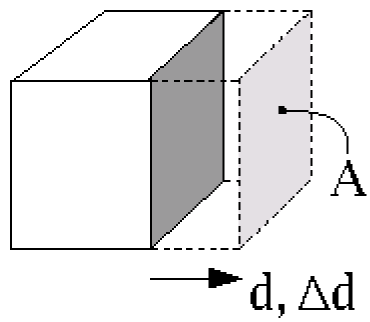
Derivation Using Conservation Of Energy (Algebra or Calculus)

Bernoulli's Equation may also be derived using the conservation of energy. While this derivation might be somewhat less intuitive than a derivation using Newton's Second Law, the conservation of energy is a helpful framework for approaching problems involving Bernoulli's Equation.

A solid object has potential energy (mgh) and kinetic energy ($mv^2/2$). Only the location and speed of the center of mass are considered, because the object is rigid.

With fluids, there is an additional energy represented by the pressure in the fluid. The mobile molecules can absorb, store and release this energy.

There are two approaches to show how the fluid pressure is equivalent to an energy per volume. In both approaches, we consider a rectangular volume with seven rigid sides and one movable side. That side is allowed to move a little, while the pressure is kept constant.



Algebra Approach	**Calculus Approach**
We start with the definition of pressure. The distance traveled determines the amount of work involved, and the change in volume. The work equals the amount of energy added to maintain the constant pressure. $P = \frac{F}{A}$ $P = \frac{F \cdot d}{A \cdot d}$ $P = \frac{W}{V}$ $P = \frac{E}{V}$	We start with the definition of work. The definitions of pressure and volume are used to set up a definite integral equation, which is then integrated. $\Delta W = F \cdot \Delta d$ $\Delta E = F \cdot \Delta d$ $\Delta E = P \cdot A \cdot \Delta d$ $\Delta E = P \cdot \Delta V$ $\int_0^E \Delta E = \int_0^V P \cdot \Delta V$ $E = P \cdot V$

where
P - static pressure (newtons per meter^2)
f - force (newtons)
A - area (meter^2)
d - distance (meter)
W - work (joules)
E - energy (joules)

Since energy is conserved, the energy for a mass of fluid at point **a** equals the energy for the same mass of fluid at point **b**. We therefore write the **energy equation** as

$$P_a V + \frac{1}{2} m {v_a}^2 + mgh_a = P_b V + \frac{1}{2} m {v_b}^2 + mgh_b$$

where
m - mass of fluid (kilograms)
v - velocity of fluid (meters / second)
g - gravitational acceleration (9.8 meters / second^2)
h - height (meters)

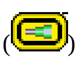
Dividing both sides by volume, and using a single density at both points, gives the **Bernoulli Equation**:

$$P_a + \frac{1}{2} \rho {v_a}^2 + \rho g h_a = P_b + \frac{1}{2} \rho {v_b}^2 + \rho g h_b$$

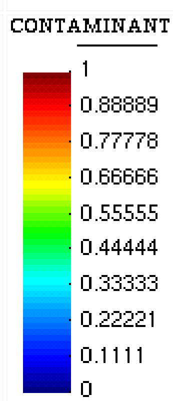
where
ρ - density of fluid (kilograms per meter^3)

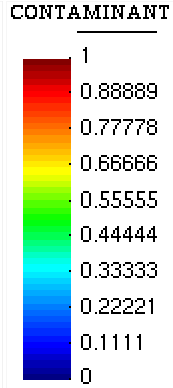
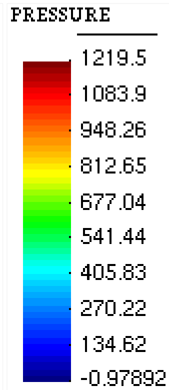
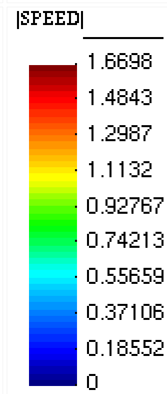
The conservation of energy is revealed in the shading above and below the pressure curve. The red shading corresponds to the energy in the pressure term of the energy equation. The blue shading corresponds to the energy in the velocity term of the energy equation. The sum of the pressure and velocity energies is always constant.

Flow Separation

When the flow separation flag () appears in the [Bernoulli's Principle Animation](), the geometry is configured in a way that would cause "flow separation" to occur in a real pipe. When flow separation occurs, the fluid no longers flow throughout the entire cross section of the pipe. Instead, the fluid flow separates away from the walls, leaving a calm region near the pipe walls. This separation tends to occur in sections of the pipe with walls that are expanding at an angle of about 7 degrees or more.

The Bernoulli's Principle Animation in this website does not model flow separation. However, you can view the following videos to see how fluid flow works when using more realistic, and much more complicated, models of fluid flow on pipes having flow separation.

Geometry	Quantity Displayed	Legend	File Size (megabytes)	File
Two chambers	**Contamination** - the color represents the amount of contamination (from dark blue at 0% to dark red at 100%)		4.8	cont_172_17_275_104.mpg

Narrow pipe widens	**Contamination** - the color represents the amount of contamination (from dark blue at 0% to dark red at 100%)		1.7	cont_218_172_433_250.mpg
Narrow pipe widens	**Pressure** - the color represents the pressure level (from dark blue at 0 to dark red at 1220)		0.7	pres_218_172_433_250.mpg
Narrow pipe widens	**Speed** - the color represents the speed (from dark blue at 0 to dark red at 1.7)		1.2	sped_218_172_433_250.mpg

Observations of the fluid flow in the videos:

- Fluid flow is much more complicated when flow separation occurs
- The majority of the fluid flows in the center of the pipe, with the boundary of that flow never steeper than an angle of 7 degrees

These videos were produced using the following excellent software tools:

- ADFC Navier-Stokes Solver developed by Grupo de Mecánica de Fluidos Computacional del Departamento de Fluidos y Calor
- GiD mesher and postprocessor developed by Centro Internacional de Métodos Numéricos en Ingeniería (CIMNE).