EEE 303 Notes: System properties

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1 Introduction

The purpose of this note is to provide a brief background and some examples on the fundamental system properties. In particular, the problems of interest have the general form,

Given a system $\mathcal{H} : \mathcal{X} \mapsto \mathcal{Y}$, $y(t) = (H[x])(t)$, determine whether it is linear, time-invariant, causal, memoryless, stable.

As a matter of notation, $y(t) = (H[x])(t)$ represents the output of the system, with input x and evaluated at time t. $y = H[x]$ denotes the output of the system with input x (the entire function). For simplicitiy, the system is referred to as "the system H" and the precise definitions of the domain and co-domain are omitted. Unless explicitly stated otherwise, the inputs and outputs are real functions defined over the real line $\mathbf{R}$.

In this context, the notations $H_1 H_2 x$, $H_1[H_2 x]$, $H_1[H_2[x]]$ are used interchangeably.

2 Linearity

2.1 Definition: A system H is said to be linear if for any scalars a, b and any inputs x_1, x_2

$$H[ax_1 + bx_2] = aH[x_1] + bH[x_2]$$

$\bigtriangledown\bigtriangledown$

Linear systems can be parametrized[1] in terms of their impulse response as follows

$$y(t) = \int_{-\infty}^{\infty} \bar{h}(t,\tau)x(\tau)d\tau$$

The impulse response $\bar{h}(t,\tau)$ is the response of H to a shifted impulse $\delta(t-\tau)$. Invoking the definition of the shift operator T_τ[2] we have

$$\bar{h}(t,\tau) = H[T_\tau \delta](t)$$

In general, $\bar{h}$ is a function of two arguments, t and τ.

2.2 Example: Let $\bar{h}(t,\tau) = \sin t \cos\tau$. Then

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} \sin t \cos\tau x(\tau)d\tau \\ &= \sin t \int_{-\infty}^{\infty} \cos\tau x(\tau)d\tau \end{aligned}$$

[1] with few exceptions

[2] The shift operator is defined by $(T_\tau[x])(t) = x(t-\tau)$, i.e., its output is the same as the input, but shifted (delayed) by τ.

For the input $x(t) = \delta(t - t_0)$,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} \sin t \cos \tau \delta(\tau - t_0) d\tau \\ &= \sin t \int_{-\infty}^{\infty} \cos t_0 \delta(\tau - t_0) d\tau \\ &= \bar{h}(t, t_0) \end{aligned}$$

▽▽

2.3 Example: Consider the system described by the differential equation

$$\dot{y}(t) \triangleq \frac{dy}{dt}(t) = -ty(t) + x(t) \ ; \quad y(t_0) = 0$$

where all signals are defined on $[t_0, \infty)$. The solution of this differential equation is

$$y(t) = \int_{t_0}^{t} e^{-(t^2 - \tau^2)/2} x(\tau) d\tau$$

To verify that this is the solution, note that $y(t_0) = 0$ and, hence, it satisfies the initial conditions. Furthermore,

$$\begin{aligned} \dot{y}(t) &= e^{-(t^2-\tau^2)/2} x(\tau)\Big|_{\tau = t} + \int_{t_0}^{t} \frac{\partial}{\partial t} e^{-(t^2-\tau^2)/2} x(\tau) d\tau \\ &= x(t) + \int_{t_0}^{t} (-t) e^{-(t^2-\tau^2)/2} x(\tau) d\tau \\ &= x(t) - ty(t) \end{aligned}$$

Therefore, the function $y(t)$ satisfies both the differential equation and the initial conditions; by the theorems on existence and uniqueness of solutions of differential equations, y is the solution.

To bring this system in the familiar form, we need to define the signals over the whole **R** and rewrite the integral with limits $-\infty, \infty$. The definition of the initial conditions at some finite time impose a constraint (and a minor inconvenience). To resolve this, we may simply ignore all input values before t_0, i.e., define the new input as $T_{t_0}[\mathcal{U}]x$. Then,

$$y(t) = \int_{-\infty}^{t} e^{-(t^2-\tau^2)/2} \mathcal{U}(\tau - t_0) x(\tau) d\tau$$

is the same as the originally defined output on the interval $[t_0, \infty)$. Notice that the new output is defined over the whole **R** whereas the original output was only defined on $[t_0, \infty)$. Although not unique, this extension is necessary to conform with the general description of linear systems. Next, the upper limit of the integration can be made ∞ by multiplying the integrant by $\mathcal{U}(t - \tau)$. Thus,

$$y(t) = \int_{-\infty}^{\infty} \mathcal{U}(t - \tau) e^{-(t^2-\tau^2)/2} \mathcal{U}(\tau - t_0) x(\tau) d\tau$$

The output is now in the standard form and we may identify the impulse response of the system as

$$\bar{h}(t, \tau) = \mathcal{U}(t - \tau) e^{-(t^2-\tau^2)/2} \mathcal{U}(\tau - t_0)$$

▽▽

2.4 Example: A typical RC circuit is described by the differential equation

$$RC\dot{y}(t) + y(t) = x(t)$$

In the ideal case, its initial condition is $y(-\infty) = 0$. Performing similar calculations as in the previous example we find that

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)/RC} \frac{x(\tau)}{RC} d\tau$$

Rewriting the integral with limits $-\infty, \infty$,

$$y(t) = \int_{-\infty}^{\infty} \mathcal{U}(t-\tau) e^{-(t-\tau)/RC} \frac{x(\tau)}{RC} d\tau$$

Thus, $\bar{h}(t,\tau) = \mathcal{U}(t-\tau)e^{-(t-\tau)/RC}/RC$. $\bigtriangledown\bigtriangledown$

Clearly, any system of the form

$$y(t) = \int_{-\infty}^{\infty} \bar{h}(t,\tau)x(\tau)d\tau \triangleq H[x](t)$$

is linear: For any scalars a, b and any input signals x_1, x_2, the linearity of integration yields

$$\begin{aligned} H[ax_1 + bx_2](t) &= \int_{-\infty}^{\infty} \bar{h}(t,\tau)[ax_1(\tau) + bx_2(\tau)]d\tau \\ &= a\int_{-\infty}^{\infty} \bar{h}(t,\tau)x_1(\tau)d\tau + b\int_{-\infty}^{\infty} \bar{h}(t,\tau)x_2(\tau)d\tau \\ &= aH[x_1](t) + bH[x_2](t) \end{aligned}$$

2.5 Example: Consider the system $y(t) = \sin(t)x(t)$. Applying the definition of linearity directly, we have

$$H[ax_1 + bx_2](t) = \sin(t)[ax_1(t) + bx_2(t)] = a\sin(t)x_1(t) + b\sin(t)x_2(t) = aH[x_1](t) + bH[x_2](t)$$

Alternatively, we may try to transform the system to the standard integral form: Since $x(t) = \int_{-\infty}^{\infty} \delta(t-\tau)x(\tau)d\tau$, we have that

$$y(t) = \int_{-\infty}^{\infty} \sin(t)\delta(t-\tau)x(\tau)d\tau$$

Hence, $\bar{h}(t,\tau) = \sin(t)\delta(t-\tau)$ is its impulse response and, of course, the system is linear. $\bigtriangledown\bigtriangledown$

2.6 Example: Consider the system $y(t) = \sin x(t)$. We suspect that it is nonlinear. So, we choose a particular test input and try to show that the linearity definition is violated. In this case, the choice is straightforward; it may not be so in general!

Let $x(t) = 1$ for all t. Then $H[ax]$ should be equal to $aH[x]$ for any a. That is $\sin(a \cdot 1) = a\sin 1$, which is obviously wrong when $a = 5$. Therefore, the system is not linear. $\bigtriangledown\bigtriangledown$

3 Time-Invariance

3.1 Definition: A system H is said to be time-invariant if it commutes with the shift operator: $HT_{t_0} = T_{t_0}H$, for all t_0. $\bigtriangledown\bigtriangledown$

In other words, the output of the cascade combination of H and T_{t_0} is the same regardless of the order that the two systems are connected.

For linear systems, an important result is that time-invariance is equivalent to

$$\bar{h}(t,\tau) = h(t-\tau)$$

that is, the impulse response is a function of the difference $t - \tau$ alone and not a function of both t and τ.

Given the impulse response $\bar{h}(t,\tau)$, one way to show that the associated system is time-varying (not time-invariant) is to choose different values for t and τ such that $t-\tau$ is the same while the value of $\bar{h}$ is different. As with any counterexample, such a choice may be easy or difficult and is highly case-dependent.

3.2 Example: Consider the system $y(t) = \sin x(t)$. Then,

$$T_{t_0}[x](t) = x(t-t_0)\ ,\quad H[T_{t_0}[x]](t) = \sin x(t-t_0)$$

On the other hand,

$$H[x](t) = \sin x(t)\ ,\quad T_{t_0}[H[x]](t) = \sin x(t-t_0)$$

Since the two are identical for any time, input signal, and shift, it follows that the system is time-invariant. ▽▽

3.3 Example: Consider the system $y(t) = \sin(t)x(t)$. Then,

$$T_{t_0}[x](t) = x(t-t_0)\ ,\quad H[T_{t_0}[x]](t) = \sin(t)x(t-t_0)$$

On the other hand,

$$H[x](t) = \sin(t)x(t)\ ,\quad T_{t_0}[H[x]](t) = \sin(t-t_0)x(t-t_0)$$

The two are obviously not identical for any time, input, and shift. A rigorous proof of this statement is often tedious and requires a suitable choice of all of the above. For example, take $x(t) = 1$ for all t, $t = 0$ and $t_0 = 1$. Then the first value is equal to zero, while the second is $\sin(-1) \neq 0$. Therefore, the system is not time-invariant.

Alternatively, we may compute the impulse response of the system which turns out to be

$$\bar{h}(t,\tau) = \sin(t)\delta(t-\tau)$$

This cannot be expressed as a function of $t-\tau$ alone. For example, let $t_1 = \tau_1 = 0$ and $t_2 = \tau_2 = 1$. In both cases, the difference is zero. However, $\bar{h}(0,0) = \sin(0)\delta(0) = 0$ while $\bar{h}(1,1) = \sin(1)\delta(0) \neq 0$. ▽▽

3.4 Example: Consider the system

$$y(t) = \int_{-\infty}^{\infty} \mathcal{U}(t-\tau)e^{-(t^2-\tau^2)/2}x(\tau)d\tau$$

Here, $\bar{h}(t,\tau) = \mathcal{U}(t-\tau)e^{-(t^2-\tau^2)/2}$ which is not a function of $t-\tau$ alone. To see this, take $t_1 = 1, \tau_1 = 0$ and $t_2 = 2, \tau_2 = 1$. Then $\bar{h}(1,0) = e^{-1/2}$ while $\bar{h}(2,1) = e^{-3/2}$. Therefore, the system is time-varying. ▽▽

3.5 Example: Consider the system

$$y(t) = \int_{-\infty}^{\infty} \mathcal{U}(t-\tau)e^{-(t-\tau)}x(\tau)d\tau$$

Here, $\bar{h}(t,\tau) = \mathcal{U}(t-\tau)e^{-(t-\tau)}$ which is a function of $t-\tau$ alone and $h(t) = \mathcal{U}(t)e^{-t}$. Therefore, the system is time-invariant.

Notice that it is customary to use t as the argument of the impulse response h, instead of defining a new symbol to denote $t-\tau$.

Alternatively, using the definition, we can establish the same property with the following sequence of computations:

$$T_{t_0}[x](t) = x(t-t_0)\ ,\quad H[T_{t_0}[x]](t) = \int_{-\infty}^{\infty} \mathcal{U}(t-\tau)e^{-(t-\tau)}x(\tau-t_0)d\tau$$

On the other hand,

$$H[x](t) = \int_{-\infty}^{\infty} \mathcal{U}(t-\tau)e^{-(t-\tau)}x(\tau)d\tau\ ,\quad T_{t_0}[H[x]](t) = \int_{-\infty}^{\infty} \mathcal{U}(t-t_0-\tau)e^{-(t-t_0-\tau)}x(\tau)d\tau$$

Writing the first integral in the form of the second, let us define the new variable of integration $\tau' = \tau - t_0$. It follows that $d\tau' = d\tau$; when $\tau \to \infty$, $\tau' \to \infty$; and when $\tau \to -\infty$, $\tau' \to -\infty$. Hence, the first integral becomes

$$\int_{-\infty}^{\infty} \mathcal{U}(t - (\tau' + t_0))e^{-(t-(\tau'+t_0))}x(\tau')d\tau'$$

which is clearly the same as the second integral. ▽▽

4 Causality

4.1 Definition: A system H is said to be causal if it does not anticipate future inputs. That is, the output at any time t depends only on the input values at or before t and is independent of the input values after t. ▽▽

While this definition is sufficient to establish causality for many simple examples, its descriptive nature makes it prone to misinterpretation. A more precise, equivalent definition makes use of the so-called truncation operator:[3]

A system H is causal iff $P_T H P_T = P_T H$ for all T.

It is interesting to observe that this definition is analogous to the definition of time-invariance. Since $P_T = P_T P_T$ (it is a projection), causality can be defined as $P_T(HP_T - P_T H) = 0$ for all T, that is, when restricted to the interval $(-\infty, T]$, the system commutes with the truncation operator. Notice, however, that this is not equivalent to $P_T H = H P_T$!

In the special case of linear systems, causality can be determined via relatively simple tests on the impulse response. For time-varying systems, causality is equivalent to $\bar{h}(t, \tau) = 0$ for $\tau > t$. For time-invariant systems this simplifies to $h(t) = 0$ for $t < 0$.

As a matter of terminology, systems whose output depends only on future values of the input are called anti-causal. For example, an anti-causal linear time-invariant system has an impulse response $h(t) = 0$ for $t \geq 0$.

4.2 Example: It follows directly from the definition that the system $y(t) = \sin(t+1)x(t-1)$ is causal but the system $y(t) = \sin(t-1)x(t+1)$ is not causal. ▽▽

4.3 Example: Scaling of the independent variable corresponds to a non-causal system. Consider the system $y(t) = x(at)$. To verify that the computation of the output requires future input information, take $t = 1$ when $a > 1$ and $t = -1$ when $a < 1$. ▽▽

4.4 Example: Consider the system

$$y(t) = \int_{-\infty}^{\infty} \sin t \cos \tau x(\tau) d\tau$$

Its impulse response is $\bar{h}(t, \tau) = \sin t \cos \tau$ which is not zero for $\tau > t$ (e.g., $t = \pi/2, \tau = 2\pi$). Hence, the system is non-causal.

From the definition,

$$\begin{aligned} P_T H P_T[x](t) &= \mathcal{U}(T - t) \sin t \int_{-\infty}^{T} \cos \tau x(\tau) d\tau \\ P_T H[x](t) &= \mathcal{U}(T - t) \sin t \int_{-\infty}^{\infty} \cos \tau x(\tau) d\tau \end{aligned}$$

The difference between the two is $\mathcal{U}(T - t) \sin t \int_T^{\infty} \cos \tau x(\tau) d\tau$ which is not identically zero for all x, t, T. ▽▽

[3] The truncation operator P_T is defined by $P_T[x](t) = x(t)$ if $t \leq T$ and $P_T[x](t) = 0$ if $t > T$.

4.5 Example: Consider the system

$$y(t) = \int_{-\infty}^{\infty} \mathcal{U}(t-\tau) e^{-(t^2-\tau^2)/2} x(\tau) d\tau$$

Its impulse response is $\bar{h}(t,\tau) = \mathcal{U}(t-\tau)e^{-(t^2-\tau^2)/2}$ which is zero for $t < \tau$. Hence, the system is causal.

Notice that this is always the case with systems described by differential equations that are solved forward in time (initial value problems). ▽▽

4.6 Example: Consider the system

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau) d\tau$$

Converting to the standard form, we find that its impulse response is $\bar{h}(t,\tau) = \mathcal{U}(t-\tau)e^{-(t-\tau)}$ which is zero for $t < \tau$. Hence, the system is causal.

We can arrive at the same conclusion by simply noticing that the upper limit of the integral is t; this implies that the computation of $y(t)$ requires values of the input only up to time t (past and present). ▽▽

Finally, to visualize the application of the second definition of causality, let us consider the last system (RC circuit) with a unit step as a test input. (Of course, equality of the two resulting outputs does not prove causality; this must hold for all possible inputs and truncation times.)

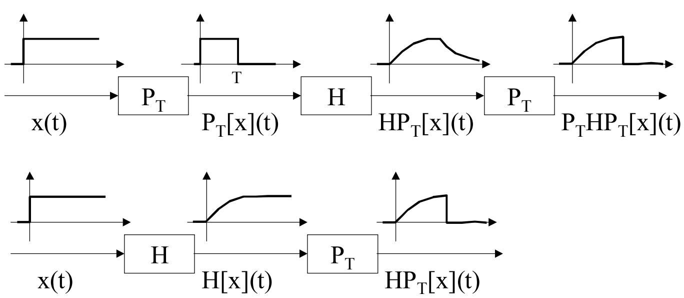

Figure 1: A demonstration of the sequence of operations involved in the causality definition.

5 Memory

5.1 Definition: A system H is called memoryless if the value of the output at time t can be determined solely from the value of the input at time t and, possibly, t. Otherwise, the system is said to have memory. ▽▽

In other words, for a memoryless system $y(t) = f(t, x(t))$ where $f(.,.)$ is a function with two arguments.

In the special case of linear systems, it follows that a memoryless system has an impulse response of the form $\bar{h}(t,\tau) = k(t)\delta(t-\tau)$, where $k(t)$ is a function of t. If, in addition, the system is time-invariant, then its impulse response is of the form $h(t) = k\delta(t)$, where k is a constant. Thus, linear memoryless systems are pure gains (ideal amplifiers).

5.2 Example: The system $y(t) = \sin(t+1)x(t)$ is memoryless; it only requires the knowledge of the value of the input at time t and the time, in order to evaluate the output. The impulse response for this system is $\bar{h}(t,\tau) = \sin(t+1)\delta(t-\tau)$. ▽▽

5.3 Example: The system $y(t) = \sin(t)x(t+1)$ has memory (is not memoryless); the output depends on values of the input other than $x(t)$. The impulse response for this system is $\bar{h}(t,\tau) = \sin(t)\delta(t-\tau+1)$. ▽▽

5.4 Example: The system $y(t) = \frac{d}{dt}x(t+1)$ (differentiator) has memory. The output depends on values of the input other than $x(t)$. A quick example of that is the functions $x(t) = t$ and $x(t) = -t$. At $t = 0$ they both have the same value but their slopes are different. For the first one, the system output would be $y(0) = 1$ and for the second, $y(0) = -1$. The impulse response for this system is $h(t) = \frac{d}{dt}\delta(t)$, the so-called unit doublet. ▽▽

5.5 Example: Consider the system

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau) d\tau$$

Converting to the standard form, we find that its impulse response is $\bar{h}(t,\tau) = \mathcal{U}(t-\tau)e^{-(t-\tau)}$ which is not of the required form. Therefore the system has memory. This is not surprising since all values of $x(t)$, from $-\infty$ to t, are needed to compute $y(t)$. ▽▽

6 Stability

6.1 Definition: A system H is said to be bounded-input, bounded-output (BIBO) stable if any bounded input produces a bounded output ▽▽

More precisely, if there exists a positive constant B_1 such that $|x(t)| \leq B_1$ for all t, then there exists a constant B_2 such that $|y(t)| \leq B_2$, $\forall t$, where $y = H[x]$. This definition allows the constant B_2 to depend on both B_1 and x. The latter case (x-dependence) can be a significant source of problems. The practically interesting case is when B_2 depends only on B_1 and satisfies certain growth conditions (e.g., B_2 grows linearly with B_1).

For linear systems, BIBO stability is equivalent to the existence of a finite constant B such that

$$\int_{-\infty}^{\infty} |\bar{h}(t,\tau)| d\tau \leq B, \quad \forall t$$

In the linear time-invariant case, this condition is further simplified to

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

that is, the impulse response h is absolutely integrable.

6.2 Remark: In the case of linear time-invariant systems described by differential equations, stability can be assessed via simple tests on the so-called characteristic equation. More specifically, a system described by the differential equation

$$a_n \frac{d^n}{dt^n} y(t) + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} y(t) + \cdots + a_1 \frac{d}{dt} y(t) + a_0 y(t) = b_m \frac{d^m}{dt^m} x(t) + \cdots + b_1 \frac{d}{dt} x(t) + b_0 x(t)$$

with $m \leq n$ is BIBO stable iff the roots of the characteristic equation

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

are in the left half-plane (have negative real parts). ▽▽

6.3 Example: The system $y(t) = \sin x(t)$ is BIBO stable since $|y(t)| \leq 1$, in fact, regardless of the bound of the input. ▽▽

6.4 Example: The system $y(t) = e^{x(t)}$ is BIBO stable since $|x(t)| \leq B_1$ implies that $|y(t)| \leq B_2 = e^{B_1}$, where we used the fact that the exponential is a non-decreasing function. ▽▽

6.5 Example: The system $y(t) = \sin t x(t)$ is BIBO stable since for $|x(t)| \leq B_1$,

$$|y(t)| \leq |\sin t||x(t)| \leq B_1$$

▽▽

6.6 Example: The system $y(t) = e^t x(t)$ is not BIBO stable. For $|x(t)| \leq B_1$, $|y(t)| \leq e^t B_1$ but e^t is not a bounded function of time. However, this inequality does not prove that the output is unbounded. To show the instability, we construct a counterexample: Let $x(t) = 1$ for all t, which is obviously bounded; then $y(t) = e^t$ which is unbounded. (That is, for any number M, there exists a time t such that $|y(t)| > M$.)

▽▽

6.7 Example: Consider the system

$$y(t) = \int_{-\infty}^{\infty} \mathcal{U}(t-\tau) e^{-(t^2-\tau^2)/2} x(\tau) d\tau$$

Its impulse response is $\bar{h}(t,\tau) = \mathcal{U}(t-\tau)e^{-(t^2-\tau^2)/2}$. Taking the integral of its absolute value we find

$$\int_{-\infty}^{\infty} |\bar{h}(t,\tau)| d\tau = e^{-t^2/2} \int_{-\infty}^{t} e^{\tau^2/2} d\tau$$

which diverges and, hence, the system is BIBO unstable (or not BIBO stable).

However, if the system starts with zero initial conditions at any finite time t_0 and a bounded input is applied over the interval $[t_0, \infty)$, the resulting output is bounded:

$$\begin{aligned} |y(t)| &\leq \int_{-t_0}^{\infty} |\bar{h}(t,\tau)||x(\tau)| d\tau \leq B_1 e^{-t^2/2} \int_{t_0}^{t} e^{\tau^2/2} d\tau \\ &\leq B_1 \underbrace{e^{-t^2/2}}_{\leq 1} \underbrace{\int_{t_0}^{1} e^{\tau^2/2} d\tau}_{C(t_0)} + B_1 \mathcal{U}(t-1) \underbrace{\int_{1}^{t} e^{-(t^2-\tau^2)/2} d\tau}_{D} \end{aligned}$$

$C(t_0)$ is a constant that depends on t_0, but is finite for any finite t_0. The term D appears only if $t > 1$ (hence the use of the unit step) and is bounded: From the identity $t^2 - \tau^2 = (t-\tau)(t+\tau)$ and for $t, \tau > 1$, the monotonicity of the exponential yields

$$e^{-(t^2-\tau^2)/2} \leq e^{-(t-\tau)\min(t+\tau)/2} \leq e^{-(t-\tau)}$$

After a simple integration we obtain $D \leq 1 - e^{-t+1} \leq 1$. Thus, the output is bounded:

$$|y(t)| \leq B_1 C(t_0) + B_1 \triangleq B_2$$

and therefore the system is stable. The constant B_2 depends critically on t_0 through $C(t_0)$ and in fact it approaches infinity as $t_0 \to -\infty$. This was manifested as the instability of the system with initial conditions at $-\infty$. This lack of uniformity of stability with respect to the initial time may arise in linear time-varying systems.

Notice that the same result can be obtained by using the impulse response of the system with finite initial time, derived in Example 2.3.

$$\int_{-\infty}^{\infty} |\bar{h}(t,\tau)| d\tau = \int_{-\infty}^{\infty} \mathcal{U}(t-\tau) e^{-(t^2-\tau^2)/2} \mathcal{U}(\tau - t_0) d\tau = \int_{t_0}^{t} e^{-(t^2-\tau^2)/2} d\tau$$

The previous derivations also show that this integral is finite and bounded by $C(t0) + 1$; hence, the system is BIBO stable

▽▽

In the last example, it should be emphasized that, although the derivations are quite straightforward, the analysis relies on experience with bounding procedures and "correct" insight on the importance of the various terms. It is included here to illustrate a type of arguments used in stability analysis. However, you are not expected to reproduce it. (at least, not until graduate school!)

6.8 Example: Consider the system

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau) d\tau$$

whose impulse response (after the usual conversion to the standard form) is $h(t) = \mathcal{U}(t)e^{-(t)}$. Taking the integral of its absolute value we find

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{0}^{\infty} e^{-t} dt = 1$$

Since this is finite, the system is BIBO stable $\bigtriangledown\bigtriangledown$

6.9 Example: Consider the system with impulse response $\bar{h}(t,\tau) = \sin t \cos \tau$. Then

$$\int_{-\infty}^{\infty} |\bar{h}(t,\tau)| d\tau \quad = \quad |\sin t| \int_{-\infty}^{\infty} |\cos \tau| d\tau$$

The last integral diverges and, whenever $\sin t \neq 0$ the integral of the absolute value of the impulse response is not finite; hence the system is unstable. $\bigtriangledown\bigtriangledown$

6.10 Remark: An important application of the concept of stability is to provide a notion of continuity of the system response, at least in terms of its gross properties. For example, a small change in the input amplitude can cause only a small change in the output amplitude. To quantify this, it is important to know the way the output bound (B_2) depends on the input bound (B_1). For linear systems, such an analysis can establish the continuity of the entire trajectory. However, a much more careful (and complicated) analysis is required for nonlinear systems. In general, drawing quick conclusions from input-output stability results can lead to misinterpretations.

To illustrate this point, let us consider the system described by the following differential equation:

$$\dot{y}(t) = y(t) - \epsilon y^3(t) + x(t)$$

Using some fairly standard tools from nonlinear systems analysis, it can be shown that this system is BIBO stable. However, small variations of the input do not necessarily produce small variations in the output. If we simulate this system with $\epsilon = 0.01$ and $x(t) = 0.01\mathcal{U}(t)$ we find that the output converges to approximately 10. On the other hand, a simulation with $x(t) = -0.01\mathcal{U}(t)$ shows that the output converges to approximately -10. Moreover, the output will still converge to roughly the same values even if the input amplitude is reduced. This example shows that, in general, even though the system may be BIBO stable, the sensitivity of the output to small variations in the input can be arbitrarily large.

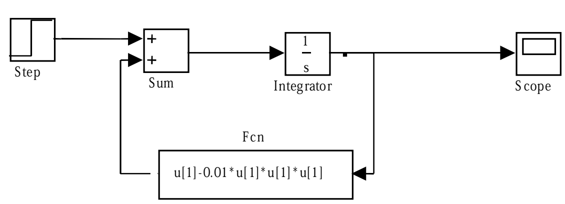

Figure 2: Simulink model of a nonlinear system.