Review of LEAD/LAG Compensator Design

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1. LEAD (Nyquist/Bode)

- **Required phase lead**

$\phi = PM_{desired} - PM_{actual} + \phi_{safety}$

PM: Phase margin, $\phi_{safety} \sim 5^o - 15^o$. If $\phi > 75^o$, then more than one lead elements are required.

- **Ratio z/p** $\frac{z_0}{p_0} = \frac{1-\sin\phi}{1+\sin\phi}$
- **New crossover:** Place gain crossover at $\sqrt{z_0 p_0}$.

Case 1: Leave low frequencies unaffected

1. Find w_m such that $|G(jw_m)| = \sqrt{z_0/p_0}$, where *G(s)* is the t.f. of the uncompensated system.
2. With w_{GC} the old gain crossover, check $-\angle G(jw_m) + \angle G(jw_{GC}) \cong \phi_{safety}$
   If greater, or much smaller, adjust ϕ_{safety} in the required lead expression (top) and iterate.
3. **Pole-zero calculations**

$$z_0 = w_m \sqrt{\frac{1-\sin\phi}{1+\sin\phi}}, \quad p_0 = w_m \sqrt{\frac{1+\sin\phi}{1-\sin\phi}}$$

4. **Implementation** $C(s) = \frac{p_0(s+z_0)}{z_0(s+p_0)}$

Case 2: Leave high frequencies unaffected

1. Find w_m such that $|G(jw_m)| = \sqrt{p_0/z_0}$, where *G(s)* is the t.f. of the uncompensated system.
2. With w_{GC} the old gain crossover, check $-\angle G(jw_m) + \angle G(jw_{GC}) \cong \phi_{safety}$
   If greater, or much smaller, adjust ϕ_{safety} in the required lead expression (top) and iterate.
3. **Pole-zero calculations**: same as Case 1.
4. **Implementation** $C(s) = \frac{(s+z_0)}{(s+p_0)}$

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2. LAG (Nyquist/Bode)

1. Gain for Stability:

Find K_0 such that $K_0G(s)$ has the desired phase margin with 5^o safety. (Must be possible, otherwise lead compensation is required.) That is, find K_0, w_{GC}, s.t.:

$$|K_0 G(jw_{GC})| = 1$$
$$\angle K_0 G(jw_{GC}) + 180 = PM_{desired} + \phi_{safety}$$

NOTE: In a lead-lag design,

$$K_0 = 1, \quad w_{GC} = w_m, \quad G(s) \leftarrow \tfrac{s+z}{s+p} G(s).$$

2. Gain for Performance:

Find K_1 such that $K_1G(s)$ meets the low-frequency performance specifications (steady-state error, disturbance attenuation, etc.)

NOTE: In a lead-lag/loop shaping design the original *G(s)* is usually selected to satisfy the low frequency specifications, so

$K_1 = p_{Lead} / z_{Lead.}$

3. Pole-zero calculations: Find z_1, p_1 such that

$$\frac{z_1}{p_1} = \frac{K_1}{K_0}, \quad z_1 < \frac{w_{GC}}{10}, \quad p_1 > w_d$$

w_d: the largest frequency for the performance specs.

4. Implementation: $C(s) = K_0 \frac{(s+z_1)}{(s+p_1)}$

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3. LEAD (Root-Locus)

1. Desired Poles: Select the location of the Dominant closed-loop pole-pair, s_1, s_1*. (Usually from overshoot, settling, rise-time specs.)

2. Angle Contributions: Find the angle contributions of the open-loop t.f. at s_1.

$$\sum\theta_z - \sum\theta_p$$

3. Required Lead: Use the argument criterion:

$$\theta_{z0} - \theta_{p0} + \sum\theta_z - \sum\theta_p = -180 \quad (K > 0)$$

4. Pole-zero calculations: Choose p_0 such that $\theta_{p0} > 5^o$, or more. Calculate z_0 from the required lead (3).

5. Gain calculations: Compute

$$K_0 = \frac{|s_1 + p_0|}{|s_1 + z_0||G(s_1)|} \quad (K > 0)$$

Check compensated root-locus for stability.

(More than one lead may be required; also, z_0 should not be too close to the dominant pair due to overshoot amplification; if so, adjust s_1 and iterate.)

6. Implementation: $C(s) = K_0 \dfrac{(s+z_0)}{(s+p_0)}$

4. LAG (Root-Locus)

1. Desired Poles: Select the location of the Dominant closed-loop pole-pair, s_1, s_1*. (Usually from overshoot, settling, rise-time specs. Must be achievable with pure gain, otherwise lead compensation is required)

Allow for some safety margin!

2. Gain for stability: Find K_0 such that $1+K_0G(s)$ has a root at s_1.

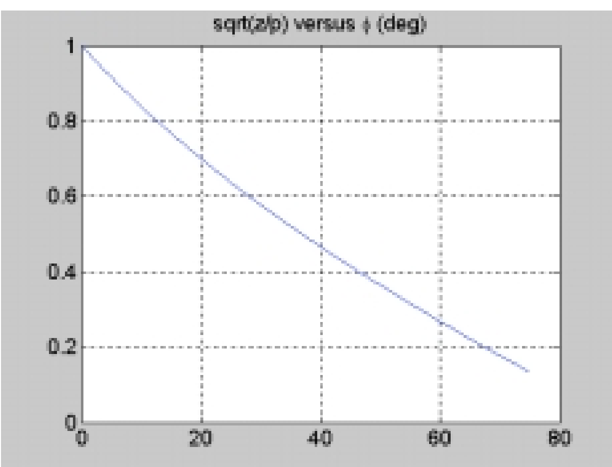
3. Gain for performance: Find K_1 such that the closed-loop of $K_1G(s)$ meets the low-frequency performance specs (steady-state error, disturbance attenuation, etc.).

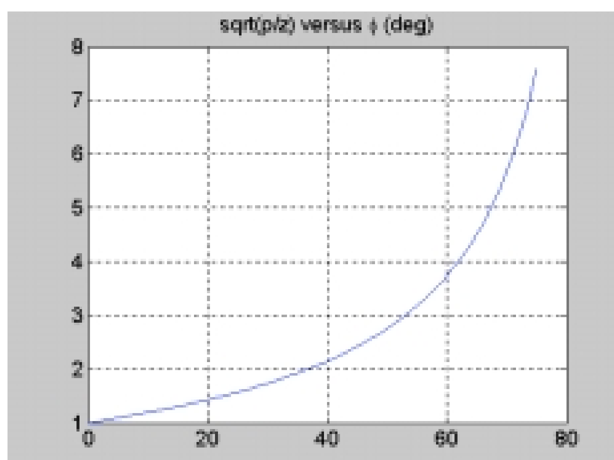
4. Pole-zero calculations: Find z_1, p_1 such that

$$\frac{z_1}{p_1} = \frac{K_1}{K_0}$$

Choose z_1 small with respect to $Re(s_1)$ and calculate p_1 from the above equation. For a largest frequency of the low frequency specs w_d, it must be $p_1 > w_d$; if not, select a new dominant pair and iterate.

5. Implementation: $C(s) = K_0 \dfrac{(s+z_1)}{(s+p_1)}$





5. CROSSOVER-BASED LEAD-LAG (Nyquist/Bode)

1. LEAD

- **Pick w_m:** Select the crossover frequency (observing possible constraints from RHP poles/zeros).

$$\phi = PM_{desired} - PM_{actual} + \phi_{safety}$$

PM: Phase margin, actual = PM of *G(s)* at w_m. $\phi_{safety} \sim 5^o - 15^o$ (Needed in lead-lag designs only.) If $\phi > 75^o$, then more than one lead elements are required.

- **Ratio *z/p*** $\frac{z_0}{p_0} = \frac{1-\sin\phi}{1+\sin\phi}$
- **Crossover:** Place gain crossover at $\sqrt{z_0 p_0} = w_m$.
- **Pole-zero calculations:**

$$z_0 = w_m \sqrt{\frac{1-\sin\phi}{1+\sin\phi}}, \quad p_0 = w_m \sqrt{\frac{1+\sin\phi}{1-\sin\phi}}$$

- **Plant gain:** Compute $g = |G(jw_m)|$, where *G(s)* is the t.f. of the uncompensated system.
- **Implementation:** $C(s) = \frac{1}{g}\sqrt{\frac{p_0}{z_0}}\frac{(s+z_0)}{(s+p_0)}$

2. LAG (of Lead-Lag)

- **Preserve the crossover w_m**
- **Implementation:** $C(s) = \sqrt{\frac{w_m^2 + p_1^2}{w_m^2 + z_1^2}}\frac{(s+z_1)}{(s+p_1)}$
- **Select z_1, p_1 such that:**
  - $\tan^{-1}\frac{w_m}{z_1} - \tan^{-1}\frac{w_m}{p_1} = -\phi_{safety}$
  - Satisfy low frequency specs.
- **Simplified design:** $C(s) = K_1 \frac{(s+z_1)}{(s+p_1)}$

$$p_1 << w_m,$$

where, $K_1 = \sqrt{\frac{1}{1 + z_1^2 / w_m^2}} \cong 1$

$$\tan^{-1}\frac{w_m}{z_1} = \frac{\pi}{2} - \phi_{safety}$$

Also valid for $p_1 = 0$. Here, adjust ϕ_{safety} (and possibly PM) to meet low frequency specs.

6. PID

1. Choose pseudo-differentiator pole one decade faster than intended bandwidth.

2. Augment plant transfer function with $\frac{1}{s(\tau s+1)}$

3. Select the PID zeros as $K(s+z)^2$ with K and z to achieve the desired R/L or N/B objectives.

4. Convert controller transfer function to standard PID form: $C_{PID}(s) = K_p + \frac{K_i}{s} + \frac{K_d s}{(\tau s+1)}$

7. PID (Ziegler-Nichols)

1. From step response data: R = effective slope (e.g., 10%-90%), L = delay (Lag, dead-time).

	P	PI	PID
Kp	1/RL	0.9/RL	1.2/RL
Ki	-	$0.27/RL^2$	$0.6/RL^2$
Kd	-	-	0.5/R

2. Experimentally, based on ultimate sensitivity: Ku = ultimate gain, Pu = ultimate period.

	P	PI	PID
Kp	0.5Ku	0.45Ku	0.6Ku
Ki	-	0.54Ku/Pu	1.2Ku/Pu
Kd	-	-	0.075KuPu

Note: Z-N tunings are such that the ideal PID (with $\tau = 0$) has a double zero, i.e., $Kp^2 = 4KiKd$.