Chapter 4

Stability, Controllability and Observability

4.1 Introduction

This chapter contains a discussion on some fundamental system properties. Stability, from a geometric point of view, is related to the properties of system trajectories around an equilibrium point. Elementary Lyapunov techniques are employed to analyze and quantify the stability of a linear system. Controllability is another geometric property of a system, describing the ability to "drive" the system states to arbitrary values through the control input. Its dual notion of observability describes the ability to infer the system states given output measurements in an interval. An elegant analysis of these structural properties is presented using vector space methods.

4.2 Stability

Consider a differential equation $\dot{x} = f(x)$. An **equilibrium point** is a vector x_e such that $f(x_e) = 0$. In other words, if $x(t_0) = x_e$, then $x(t) = x_e$, for all $t \geq t_0$. Obviously, linear systems ($\dot{x} = Ax$) always have 0 as an equilibrium point. Geometric stability (in the sense of Lyapunov) is defined with respect to such equilibrium points: *"Given any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $\|x(t_0) - x_e\| < \delta \Rightarrow \|x(t) - x_e\| < \epsilon$,* $\forall t \geq t_0$. That is, the solution will stay close to the equilibrium for any initial condition "sufficiently close" to it. Further, an equilibrium is said to be **asymptotically stable** if it is stable and $x(t) \to x_e$ as $t \to \infty$ for any initial condition sufficienly close to the equilibrium. In addition, if the convergence is exponential ($\|x(t) - x_e\| \leq K(x_0)e^{-a(t-t_0)}$, where $a > 0$ and $K(x_0)$ is a constant that depends on the initial conditions) the equilibrium is exponentially stable.

This notion of stability is different from the input-output (operator) stability where a system is L-stable if any input in L produces an output in L. Here L is a vector space, e.g., bounded functions, energy functions etc. The input-output stability is associated with concepts like operator gains, approximation and robustness and is useful in describing performance specifications. On the other hand, Lyapunov stability is suitable to describe convergence properties and provides a more appealing computational framework. While in the case of linear time invariant systems the two stability notions are closely related, their differences become more pronounced (and technically involved) in the general nonlinear case.

The basic Lyapunov analysis begins with a positive definite function of the states, interpreted as the energy stored in the system, e.g., $V(x) = x^\top Px$ where $P = P^\top > 0$. A sufficient condition for the asymptotic stability (stability) of the zero equilibrium is that the derivative of thus function along the trajectories of the system ($dV/dt = (\partial V \partial x)\dot{x}$) is negative definite (semi-definite). This can be viewed as a condition on the energy dissipation within the system. It is also a necessary condition in the sense that if an equilibrium is asymptotically stable, then there exists a Lyapunov function with the above properties. [1] In general, it is difficult to construct such a function. Nevertheless, in the case of linear systems the Lyapunov functions are quadratic making their computation a straightforward exercise in matrix algebra.

To demonstrate the application of Lyapunov analysis, let us consider the system $\dot{x} = Ax$ and the function $V = x^\top Px$. The derivative of V along the trajectories of the system is computed as follows:

$$\dot{V} = \frac{\partial V}{\partial x}\frac{dx}{dt} = x^\top(A^\top P + PA)x$$

Next, suppose that the matrix $Q = -(A^\top P + PA)$ is positive definite. Then, the following inequalities can be established:

$$\dot{V} \le -\lambda_{min}(Q)\|x\|^2 \le -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}V$$

Using the so-called comparison principle, it now follows that $V(t) \le V(0)e^{-at}$ where $a = \frac{\lambda_{min}(Q)}{\lambda_{max}(P)}$. In turn, this implies that for any initial conditions, the system trajectories converge to the zero equilibrium exponentially fast. These derivations illustrate the essense of Lyapunov analysis for linear systems and are made precise by the following theorem:

4.2.1 Theorem: *Given a matrix $A \in \mathbf{R}^{n\times n}$ the following statements are equivalent:*

1. *The eigenvalues of A have strictly negative real parts (A is* **Hurwitz***).*
2. *There exist positive constants K, a such that $\|e^{At}\| \le Ke^{-at}$.*
3. *There exists some $Q = Q^\top > 0$ such that $A^\top P + PA = -Q$ has a unique solution for P and it is positive definite.*
4. *For any $Q = Q^\top > 0$, there exists a unique P such that $A^\top P + PA = -Q$ and this P is positive definite.*

$\nabla\nabla$

Notice that, for a Hurwitz matrix A, not every positive definite P produces a positive definite Q; only the converse holds. The equation $A^\top P + PA = -Q$ is referred to as **Lyapunov** equation. It is linear in P and can be solved as a system of linear equations. In fact, this equation has a unique solution (positive definite or not) iff any two eigenvalues of P satisfy $\lambda_i + \lambda_j \ne 0$. From a system theoretic viewpoint, a more interesting property of Lyapunov equations is that for a Hurwitz A, their solution has the form

$$P = \int_0^\infty e^{A^\top t}Qe^{At}dt$$

The last expression is extremely important for its analytical value. Among other applications, it allows an easy computation of controllability and observability Gramians as solutions of linear Lyapunov equations. These are an integral component of general model order reduction algorithms.

Lyapunov equations play an important role in several recent results on the design of control systems via numerical optimization. For example, consider the (intermediate) problem where given a matrix A we would like to estimate the exponential rate of decay of the states to zero. This can be found as the real part of the eigenvalue of A closest to the jw-axis. However, eigenvalues are nonlinear functions of the entries of A and are not suitable objectives for any (additional) optimization. Alternatively, we can ask to find the matrix Q that maximizes the ratio $a = \frac{\lambda_{min}(Q)}{\lambda_{max}(P)}$. As previously shown this ratio provides an estimate of the rate of decay of the states ($V(x) \sim \|x\|^2$). It can be shown that the optimal Q for this purpose is the identity. This problem can be cast as the optimization of a convex objective subject to convex constraints (linear matrix inequalities), and its solution can be obtained with numerically efficient and reliable algorithms. The value

[1] While these statements are adequate to describe the general picture, they lack the necessary precision for use with general nonlinear systems.

of this approach lies in its ability to handle cases where the matrix A is itself a convex function of other parameters. A simple example of that is to find a single Lyapunov function, if it exists, that has a negative definite derivative for two systems, i.e.,

$$P = P^\top > 0 : \quad A_1^\top P + PA_1 \leq -I \quad ; \quad A_2^\top P + PA_2 \leq -I$$

The existence of such a P would imply the stability of a system whose matrix A undergoes arbitrarily fast transitions between the values A_1 and A_2. This type of problems arises in the analysis and design of gain-scheduled control systems.

For linear systems it is straightforward to show that exponential stability of the zero equilibrium (A being Hurwitz) also implies the input-output stability (in a BIBO or energy sense) of the system $[A, B, C, D]$, for any B, C, D. The converse is not always true unless some additional conditions are imposed, e.g., controllability and observability. Furthermore, a somewhat similar statement is valid in a general nonlinear setting but with significantly more involved technical conditions.

4.3 Controllability and Observability

The fundamental controllability problem is associated with the question whether an input can be found such that the system states can be steered from an initial value x_0 to any final value x_1 in a given time interval. In general, the answer to this question depends on the time interval. This induces different notions of controllability (uniform, instantaneous, differential) all of which become equivalent for linear time invariant systems. In the following discussion, only this simpler case is considered.

4.3.1 Definition: *For the system $\dot{x} = Ax + Bu$, $x(0) = x_0$, the state x_0 is said to be controllable if for any x_1 there exists a time $t_1 > 0$ and an input $u_{[0,t_1]}$ such that $x(t_1) = x_1$. Furthermore, we say that (A, B) is* **completely controllable** *(c.c.) if every x_0 is controllable.* ▽▽

Using the general solution of the linear differential equation, it is easily shown that

- (A, B) is c.c. iff every x_0 can be steered to the origin at t_1.
- (A, B) is c.c. iff every x_1 can be reached from the origin at t_1.

Another important result can be obtained in terms of the so-called **controllability Gramian**

$$M(t_0, t_1) = \int_{t_0}^{t_1} e^{A(t_1-\tau)} BB^\top e^{A^\top (t_1-\tau)} d\tau$$

4.3.2 Theorem: *(A, B) is c.c. iff $M(t_0, t_1)$ is positive definite for some $t_1 > t_0$.* ▽▽

Proof: (if) Since M is positive definite, it is nonsingular and therefore its inverse is well-defined. Define the input as

$$u(\tau) = B^\top e^{A^\top (t_1-\tau)} M^{-1}(t_0, t_1)[x_1 - e^{A(t_1-t_0)} x_0]$$

Then, by direct substitution, it follows that $x(t_1) = x_1$.

(only if) This is a more involved argument and we will need the following intermediate result:

Lemma: *Let $F : [t_0, t_1] \mapsto \mathbf{C}^{n\times m}$ be a continuous matrix valued function. F has linearly independent rows iff the Gramian $G(t_0, t_1) = \int_{t_0}^{t_1} F(\tau)F^H(\tau)d\tau$ is positive definite.*

A fairly standard approach to this type of proofs is to consider a quadratic form $v^H Gv$ where v is an arbitrary vector. Then $v^H Gv = 0 \Leftrightarrow v^H F(t) \equiv 0$.

Returning to our proof, let $F(t) = e^{A(t_1-t)}B$ and suppose that M is not positive definite. (Notice that, still, $M \geq 0$.) Then, there exists $v \neq 0$ such that $v^\top Mv = 0$. By virtue of the Lemma, this implies $vF(t) \equiv 0$. Let $x_0 = e^{A(t_0-t_1)}v$. By the c.c. assumption, there exists an input u such that it steers x_0 to the origin in the interval $[t_0, t_1]$. It now follows that $0 = v + \int_{t_0}^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau$ and taking the inner product with v we get that $0 = v^\top v + \int_{t_0}^{t_1} v^\top e^{A(t_1-\tau)} Bu(\tau)d\tau$. But the second term is zero, leading to the conclusion $v^\top v = 0$ that contradicts the assumption that M is singular. □□

An elegant re-statement of the last property comes with the interpretation of controllability as a rank condition on the linear map that relates inputs to final states. That is, consider the linear map

$$\mathcal{A} : L_2 \mapsto \mathbf{R}^n;\ \xi = \mathcal{A}[u] = \int_{t_0}^{t_1} e^{A(t-\tau)} Bu(\tau)d\tau$$

where L_2 is the space of square integrable functions on $[t_0, t_1]$. L_2 is a Hilbert space with inner product $< x, y >= \int_{t_0}^{t_1} x^\top(\tau)y(\tau)d\tau$. In this framework, controllability simply means that $\mathcal{A}$ is onto. In view of the four fundamental subspaces associated with linear maps, this condition translates into $\mathcal{N}(\mathcal{A}^*) = \{0\}$. Since the range of $\mathcal{A}$ is finite dinensional and therefore closed, this would imply that $\mathcal{A}$ is onto. The adjoint map $\mathcal{A}$ is defined in terms of the inner products of the domain and co-domain: For any $u \in L_2$ and $\xi \in \mathbf{R}^n$, we should have

$$< \xi, \mathcal{A}[u] >_{\mathbf{R}^n} = < \mathcal{A}^*[\xi], u >_{L_2}$$

It is now easy to verify that the adjoint map is a multiplier mapping vectors into functions:

$$\mathcal{A}^*[\xi](t) = B^\top e^{A^\top(t_1-t)}\xi$$

This equation describes immediately one of the equivalent controllability conditions: $\mathcal{N}(\mathcal{A}^*) = \{0\}$ iff the columns of $B^\top e^{A^\top(t_1-t)}$ (rows of $e^{A(t_1-t)}B$)are linearly independent functions of time. We may also observe that $\mathcal{N}(\mathcal{A}^*) = \mathcal{N}(\mathcal{A}\mathcal{A}^*)$. The operator $\mathcal{A}\mathcal{A}^*$ maps $\mathbf{R}^n$ to $\mathbf{R}^n$, i.e., it is a matrix. It is easy to verify that $\mathcal{A}\mathcal{A}^* = M(t_0, t_1)$ from which the previous theorem follows.

In this vector space framework it is easy to answer an additional question: "What is the minimum norm input that transfers x_0 to x_1?" (Notice that $\mathcal{A}$ may be onto but it cannot be 1-1 since the domain is infinite dimensional while the range is finite dimensional.) The answer to this question is immediate from the Classical Projection Theorem:

$$u_{opt} = \mathcal{P}_{\mathcal{N}(\mathcal{A})^\perp}[u_1]$$

where u_1 is a solution to $\mathcal{A} = \xi$. Under the c.c. condition that $M(t_0, t_1)$ is invertible, the solution can be written directly, without computing an intermediate u_1:

$$u_{opt}(t) = \mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}\xi = B^\top e^{A^\top(t_1,t)} M^{-1}(t_0, t_1)\xi$$

Also recall that all solutions are now parametrized as $u = u_{opt} + \mathcal{N}(\mathcal{A})$, where the null space is composed of functions orthogonal to the range of $\mathcal{A}^*$.

The controllability condition in terms of the Gramian is an extremely useful tool both for analysis and for numerical computations. Still, other equivalent conditions may be more convenient to check, depending on the problem at hand. Staying with the Gramian for the moment, a simple computation establishes the intuitive result: If (A, B) is c.c. in an interval $[t_0, t_1]$ it is also c.c. in any interval $[t_0, t_2]$ for any $t_2 > t_1$.

$$\begin{aligned} M(t_0, t_2) &= \int_{t_0}^{t_1} e^{A(t_2-\tau)} BB^\top e^{A^\top(t_2-\tau)} d\tau + \int_{t_1}^{t_2} e^{A(t_2-\tau)} BB^\top e^{A^\top(t_2-\tau)} d\tau \\ &= e^{A(t_2-t_1)} M(t_0, t_1) e^{A^\top(t_2-t_1)} + M(t_1, t_2) \end{aligned}$$

Since the last term is always positive semidefinite and the matrix exponential is always nonsingular, the desired result follows. Furthermore, for time invariant systems, a simple change of variables under the integral shows that controllability is equivalent to the nonsingularity of $M(0, t)$. Combining the two, we get that

- (A, B) is c.c. iff $M(0, t) > 0$, $\forall t \geq t_1$, for some $t_1 > 0$.

Through a different line of arguments, it can be shown that in the time invariant case controllability on an interval also implies controllability on a smaller interval. This last statement makes controllability easier to check for Hurwitz A, since $M(0,\infty)$ can be computed by solving a simple Lyapunov equation:

$$AM + MA^\top = -BB^\top$$

We have already seen that this computation produces a positive definite solution M when $BB^\top > 0$. The same is true if (A, B) is controllable. When A is not Hurwitz, the statement is still valid for any finite interval. However, the Gramian does not converge as $t \to \infty$ and cannot be computed as the solution of a Lyapunov equation. In this case, different techniques should be employed (e.g., splitting A into stable and anti-stable parts).

A different equivalent controllability condition is obtained through the use of Wronskians and the following two technical Lemmas:

Lemma: Let $F : [t_0, t_1] \mapsto \mathbf{R}^{n,m}$ be a matrix function of time with $(n-1)$ continuous derivatives on $[t_0, t_1]$. If the **Wronskian** $W(t) = [F(t), F^{(1)}(t), \ldots, F^{(n-1)}(t)]$ has rank n at some $t \in [t_0, t_1]$ then the rows of $F(t)$ are linearly independent functions of time on $[t_0, t_1]$.

Lemma: In the previous lemma, suppose that the entries of F are analytic functions of time. Then the rows of F are linearly independent on $[t_0, t_1]$ iff the Wronskian W has rank n almost everywhere in $[t_0, t_1]$.

The second lemma is more useful to our case since the functions involved are polynomials and exponentials. Thus, evaluating the Wronskian at $t = 0$ we obtain the following controllability condition

- (A, B) is c.c. iff the **controllability matrix** $Q_c = [B, AB, A^2B, \ldots, A^{n-1}B]$ has rank n, or equivalently $Q_cQ_c^\top$ is invertible.

This condition is one of the easiest to check given A, B and holds regardless of the stability properties of A. However, it is not always the most suitable. Other conditions are obtained via transformations of the above basic conditions; they are summarized in the following theorem.

4.3.3 Theorem: *For the linear time invariant system $\dot{x} = Ax + Bu$, the following statements are equivalent:*

- *(A, B) is completely controllable.*
- *The controllability Gramian satisfies $M(0, t) > 0$ for all $t > 0$.*
- *The controllability matrix Q_c has rank n ($Q_cQ_c^\top > 0$).*
- *The rows of $e^{At}B$ are linearly independent functions of time.*
- *The rows of $(sI - A)^{-1}B$ are linearly independent functions of s.*
- *$r([A - \lambda I, B]) = n$ for all λ (suffices to check only the eigenvalues of A).*
- *$v^\top B = 0$ and $v^\top A = \lambda v^\top \Rightarrow v = 0$ (Popov-Belevich-Hautus test).*
- *Given any set Γ of numbers in $\mathbf{C}$ there exists a matrix K such that the spectrum of $A + BK$ is equal to Γ; note that if A, B, K are real, Γ should be symmetric about the real line. (Pole placement condition)*

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Analogous results can be obtained for the observability problem. Here, the fundamental question is posed as follows: Given the system $\dot{x} = Ax$, $y = Cx$ with $x(t_0) = x_0$ and measurements $y(t)$ in an interval $[t_0, t_1]$, find x_0. If this is possible, x_0 is said to be observable and if every x_0 is observable then (A, C) is completely observable (c.o.). In terms of operators in vector spaces, the map from initial conditions to the output is given by a multiplier:

$$\mathcal{B} : \mathbf{R}^n \mapsto L_2; \quad \mathcal{B}[x_0] = Ce^{A(t-t_0)}x_0$$

In general, this is a least squares problem; but in our noise-free case a solution exists and the main issue is the uniqueness of the solution. In other words, we seek to determine whether the operator $\mathcal{B}$ is 1-1. Its adjoint is an integral operator

$$\mathcal{B}^* : L_2 \mapsto \mathbf{R}^n; \quad \mathcal{B}^*[y] = \int_{t_0}^{t_1} e^{A^\top(\tau - t_0)} C^\top y(t) d\tau$$

and $\mathcal{N}(\mathcal{B}) = \mathcal{N}(\mathcal{B}^*\mathcal{B})$. The last operator is again a matrix mapping $\mathbf{R}^n$ to $\mathbf{R}^n$. It is referred to as the **observability Gramian**:

$$N(t_0, t_1) = \int_{t_0}^{t_1} e^{A^\top(\tau - t_0)} C^\top C e^{A(\tau - t_0)} d\tau$$

If nonsingular, the unique solution for x_0 can be found with the usual least squares formula

$$x_0 = N^{-1}(t_0, t_1) \int_{t_0}^{t_1} e^{A^\top(\tau - t_0)} C^\top y(\tau) d\tau$$

The observability Gramian is similar to the controllability Gramian with A, B being changed to $-A^\top, C^\top$. The remaining difference is in that M contains t_1 while N contains t_0. This can be fixed by pre- and post-multiplying with $e^{A(t_0 - t_1)}$ and its transpose. The latter being invertible does not change the rank properties of the matrix. This observation suggests a deeper duality between controllability and observability.

4.3.4 Definition: *Given the system $[A, B, C, D]$, the adjoint system is defined as $[-A^\top, C^\top, B^\top D^\top]$.* [2]

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4.3.5 Theorem: *(Duality)*

- *$[A, B, C, D]$ is c.c. iff $[-A^\top, C^\top, B^\top, D^\top]$ is c.o.*
- *$[A, B, C, D]$ is c.o. iff $[-A^\top, C^\top, B^\top, D^\top]$ is c.c.*

$\nabla\nabla$

Thus, all our controllability conditions can be readily extended to the observability case. In particular, the **observability matrix** is now defined as

$$Q_o = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

instead of its transpose, and complete observability is equivalent to the invertibility of $Q_o^\top Q_o$. Furthermore, whenever applicable, the infinite-interval Gramian is now found as the solution of the Lyapunov equation

$$A^\top N + NA = -C^\top C$$

As a final remark, we should comment on the difference between the controllability Gramian and the controllability matrix. From the previous analysis they both provide necessary and sufficient conditions for controllability. The Gramian represents an integral condition while the matrix is associated with instantaneous (or, better, differential) properties. While equivalent for time-invariant systems, they produce different results with time-varying systems. Furthemore, their difference has an impact even in the time-invariant case. The Gramian arises whenever operator properties are concerned (optimal linear quadratic regulator, system reduction with a system gain criterion); the matrix appears in cases where the state-space representation is altered (pole-placement control, conversion to canonical forms, structural versions of model reduction).

[2] As usual, complex-conjugate transpose should be used in the complex case.

4.4 Optimal Linear Quadratic Regulators and Observers

Controllability and observability are central in the design of control systems since, among other properties, they guarantee the existence of a stabilizing controller. Such a controller can be designed as

$$u = Kx$$

where K is such that $A+BK$ is Hurwitz (e.g., via the pole-placement equations). If the states of the system cannot be measured, then an observer based controller can achieve stabilization by setting $u = -K\hat{x}$ where $\hat{x}$ is the state of an observer

$$\dot{\hat{x}} = A\hat{x} + Bu + L(\hat{y} - y) \ ; \quad \hat{y} = C\hat{x} + Du$$

where L is such that $A + LC$ is Hurwitz (e.g., via the dual to pole-placement equations). This observer-based controller satisfies the so-called **separation principle**, stating that the closed-loop eigenvalues are the eigenvalues of $A + BK$ and the eigenvalues of $A + LC$. Even though stabilization is seemingly easy to achieve, the design of such a controller with desirable closed-loop properties (overshoot, bandwidth etc.) is more obscure.

In the 60's the state-space approach became very popular with the application of optimal control theory. The optimal linear quadratic regulator (LQR) problem is defined as the design of a control input that minimizes an integral cost of the form

$$J = \int_0^\infty [x^\top(t)Qx(t) + u^\top(t)Ru(t)]dt$$

subject to the constraint $\dot{x} = Ax + Bu$, $x(0) = x_0$, and with $Q \geq 0$, $R > 0$. An important property of this design is that regardless of the choice of Q, R, the resulting controller is a stabilizing one. Moreover, it is always well-behaved and possesses some interesting and very desirable properties in terms of sensitivity peak magnitude. The separation principle was again invoked to design a controller by coupling the LQR with a Kalman Filter (LQ-optimal observer). Unfortunately, the result did not meet the expectations. The introduction of an observer in the loop can destroy the nice properties of the state-feedback LQR and introduce undesirable high gains in parts of the loop. This was manifested as a loss of robustness since a small amount of uncertainty could cause destabilization of the closed-loop.

Subsequently, the 80's witnessed the rise of the H_∞ theory, which aimed to design controllers with optimal robustness properties. The classical notions of frequency domain peak magnitude, gain and phase margins came to the picture once again, though transformed to fit a multivariable framework. Here, instead of the Q, R matrices, the controller design parameters were multivariable transfer matrix weights. In the heart of the solution there was still an optimal LQ problem, except that now the controller and observer design equations were coupled. This is not surprising since the LQR possesses desirable induced-L_2 gain properties and robustness was formulated as the ability to tolerate perturbations of a certain induced-L_2 gain. The results were then re-derived taking a differential game approach where a similar quadratic cost is minimized with the addition of an energy-bounded disturbance being the adversary.

In the following, we present some of the fundamental properties of the LQR theory. The basic LQR solution can be derived using Pontryagin's minimum principle or the Bellman's principle of optimality. The solution has the form

$$u = Kx = -R^{-1}B^\top Px$$

where P is the positive definite solution of the so-called control algebraic Riccati equation (CARE)

$$A^\top P + PA - PBR^{-1}B^\top P + Q = 0$$

A positive definite solution to the CARE exists if (A, B) is c.c. and (A, Q) is c.o. Notice that if $Q = C^\top C$, the latter condition becomes (A, C) is c.o.[3]

[3] In general, stabilizability and detectability is sufficient, i.e., any uncontrollable and/or unobservable modes should be stable.

Under these conditions, we claim that the control law $u = Kx$ is a stabilizing one. For this, consider the Lyapunov function candidate $V = x^\top Px$. Taking its derivative along the system trajectories, we get

$$\begin{aligned}\dot{V} &= x^\top(A^\top P + PA - 2PBR^{-1}B^\top P)x \\ &= -x^\top Qx - u^\top Ru\end{aligned}$$

Hence, $\dot{V}$ is at least negative semi-definite implying the stability of the zero equilibrium. If Q is only positive semi-definite, then an additional argument is needed to establish asymptotic stability. This is known as LaSalle's theorem which states that the solution will converge to the set where $\dot{V} = 0$. If $Q = C^\top C$ and (A, C) is observable, then the only solution of the system differential equation that satisfies $\dot{V} = 0$ is the trivial one. Hence, the states converge to zero asymptotically with time.

Another important result is related to the controller behavior in the presence of external disturbances and, in particular, input disturbances. Let us consider the system

$$\dot{x} = Ax + B(u + d)$$

with u obtained from the LQR. A similar Lyapunov argument now yields

$$\begin{aligned}\dot{V} &= x^\top(A^\top P + PA - 2PBR^{-1}B^\top P)x + 2x^\top PBd \\ &= -x^\top Qx - u^\top Ru - 2u^\top Rd \\ &= -x^\top Qx - (u + d)^\top R(u + d) + d^\top Rd\end{aligned}$$

Integrating both sides we obtain

$$\|u + d\|_{2,R}^2 \le V(0) + \|d\|_{2,R}^2$$

In other words, the gain of the closed-loop operator mapping $d \mapsto u + d$ is less than one. In terms of classical control theory, this implies that the closed-loop system has infinite gain margin and at least 60 deg. phase margin for perturbations entering at the plant input. This type of result is central in the linear quadratic theory. One of its important by-products is that by augmenting the system with suitable weights, the operator from an external input to an output of interest can be manipulated to possess desirable properties (e.g., augmentation by an integrator).

Finally, the linear quadratic observer design can be performed as the dual of LQR. Here, $L = -P_oC^\top R_o^{-1}$ and P_o is the positive definite solution of the filtering ARE (FARE)

$$P_oA^\top + AP_o - P_oC^\top R_o^{-1}CP_o + Q_o = 0$$

with $R_o > 0$, $Q_o \ge 0$ and (A, Q_o) c.c. In the classical Kalman Filter theory, Q_o, R_o are related to the output noise and state noise intensities (covariances). However, as previously mentioned, such a selection does not necessarily lead to good observer-based controllers. Instead, the H_∞ solution uses fictitious noise intensities, in addition to an augmentation with suitable filters, to shape the closed-loop sensitivities in an optimal manner.

4.5 Realization Theory Fundamentals

The basic problem in Realization Theory is to determine matrices A, B, C, D such that the state space system $[A, B, C, D]$ represents a given input-output map, specified by its impulse response or transfer function.

4.5.1 Definition:

- $[A, B, C, D]$ *and* $[\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}]$ *are zero-state equivalent (I/O equivalent) if they correspond to the same impulse response, i.e.,*

$$C(sI - A)^{-1}B + D = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}$$

- *$[A,B,C,D]$ and $[\tilde{A},\tilde{B},\tilde{C},\tilde{D}]$ are algebraically equivalent if they are of the same dimension and they are related by a similarity (coordinate) transformation, i.e., there exists an invertible matrix T such that*

$$\tilde{A} = T^{-1}AT \ ; \quad \tilde{B} = T^{-1}B \ ; \quad \tilde{C} = CT \ ; \quad \tilde{D} = D$$

- *$[A,B,C,D]$ is reducible (non-minimal) if there exists a zero-state equivalent representation of smaller state-space dimension; otherwise $[A,B,C,D]$ is irreducible or minimal.*

▽▽

Notice that algebraic equivalence implies zero-state equivalence. The converse may not be true even if both systems have the same state-space dimension.

4.5.2 Theorem: *Controllability, observability and stability are invariant under similarity transformations.*

▽▽

Proof: Direct computations establish the following relations between the Gramians and State Transition matrices of two similar realizations:

$$\tilde{M} = T^{-1}MT^{-\top} \quad ; \quad \tilde{N} = T^{\top}NT \quad ; \quad e^{\tilde{A}t} = T^{-1}e^{At}T$$

□□

In other words, algebraically equivalent realizations have identical controllability, observability and stability properties. An implication of that is that the following two systems cannot be algebraically equivalent even though they are zero-state equivalent and they have the same state-space dimension:

$$\left[\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (1,1), 0\right] \quad ; \quad \left[\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, (1,0), 0\right]$$

Notice that the first system is c.o. but not c.c. while the second is c.c. but not c.o.

Some properties of minimal realizations are described by the following theorem. They make use of the so-called **Hankel matrix**

$$H = Q_oQ_c = \begin{pmatrix} CB & CAB & CA^2B & \dots \\ CAB & CA^2B & & \\ CA^2B & & \ddots & \\ \vdots & & & \end{pmatrix}$$

4.5.3 Theorem:

- *$[A,B,C,D]$ is minimal iff it is c.c. and c.o.*
- *$[A,B,C,D]$ is minimal iff its Hankel matrix has rank n.*
- *Suppose $[A,B,C,D]$ is minimal and let $[\tilde{A},\tilde{B},\tilde{C},\tilde{D}]$ be a zero-state equivalent realization. Then $[\tilde{A},\tilde{B},\tilde{C},\tilde{D}]$ is minimal iff it is algebraically equivalent to $[A,B,C,D]$. In such a case, the similarity transformation relating the two can be computed as*

$$T = Q_c\tilde{Q}_c^{\top}(\tilde{Q}_c\tilde{Q}_c^{\top})^{-1} \quad ; \quad T^{-1} = (\tilde{Q}_o^{\top}\tilde{Q}_o)^{-1}\tilde{Q}_o^{\top}Q_o$$

▽▽

In dealing with nonminimal realizations the following result is important, providing both the theoretical framework and a computational approach to construct I/O equivalent minimal realizations.

4.5.4 Theorem: *(Kalman Canonical Decomposition) Consider the system $\dot{x} = Ax + Bu$, $y = Cx$. There exists a coordinate transformation T such that $x = T\bar{x}$ and*

$$\bar{A} = \begin{pmatrix} \bar{A}_{11} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{22} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{33} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{44} \end{pmatrix} \quad ; \quad \bar{B} = \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad \bar{C} = (\bar{C}_1, 0, \bar{C}_3, 0)$$

with the following properties:

1. *$[\bar{A}_{11}, \bar{B}_1, \bar{C}_1]$ is minimal.*
2. *$[\bar{A}_{11}, \bar{B}_1, \bar{C}_1]$ and $[A, B, C]$ are I/O equivalent.*
3. *The states $\bar{x}_2$ are c.c. but unobservable.*
4. *The states $\bar{x}_3$ are c.o. but uncontrollable.*
5. *The states $\bar{x}_4$ are uncontrollable and unobservable.*

where the state partition is compatible with the $\bar{A}$-matrix partition. Furthermore, the transformation T can be computed systematically using SVD or QR decompositions of the controllability and observability matrices: $\mathcal{R}(Q_c)$ is the controllable subspace and $\mathcal{N}(Q_o)$ is the unobservable subspace. ▽▽

Using the Kalman Canonical Decomposition a computational procedure to obtain a minimal realization is described below. This approach is not necessarily numerically efficient or reliable; nevertheless it establishes a systematic method to obtain minimal realizatons.

4.5.5 Corollary: *Given $[A, B, C, D]$:*

1. *Compute the controllability matrix Q_c and find T_1, an orthonormal basis of the columns of Q_c. (E.g., Let $Q_c = USV^\top$ and set $T_1 = U_1$, the first $r(Q_c)$ columns of U.)*
2. *Define the system $[T_1^\top AT_1, T_1^\top B, CT_1, D]$, which is a realization of the controllable subsystem and compute its observability matrix $\bar{Q}_o$.*[4]
3. *Find the matrix T_2 that is an orthonormal basis of the columns of $\bar{Q}_o^\top$. (E.g., Let $\bar{Q}_o^\top = USV^\top$ and set $T_2 = U_1$, the first $r(\bar{Q}_o)$ columns of U.)*

Then

$$[T_2^\top (T_1^\top AT_1)T_2,\ T_2^\top (T_1^\top B),\ (CT_1)T_2,\ D]$$

is an I/O equivalent minimal realization. (The order of computations can be interchanged.) ▽▽

4.6 Balanced Realizations and Model Reduction

Kalman's canonical decomposition provides the basic theory and computational algorithm to remove unnecessary states from a realization, while preserving the input-output map. Its reliance on the controllability and observability matrices makes the approach somewhat susceptible to numerical problems, as these matrices are often poorly conditioned. A more serious drawback, however, is that this reduction is based on structural properties of the system (linear independence) but without explicitly considering the quantitative aspects of the problem. In practical applications, especially when numerical computations are involved, one is rarely faced with perfectly dependent or perfectly orthogonal vectors. Moreover, a commonly encountered problem is that of a model reduction where modes that have independent but small contributions should be eliminated. With such an objective in mind, the previous algorithm is inadequate. While the SVD of the controllability matrix can give an indication on the modes that are weekly controllable, these modes

[4]Notice that this is not a similarity transformation.

cannot be immediately eliminated. The reason is that they may be strongly observable and, hence, have non-negligible contribution to the system response.

In order to deal with model reduction problems, we need a quantitative approach that correctly accounts for the strength of the contribution of each mode to the system response. We discussed one such method in the first chapter via partial fraction expansions.[5] Several general methods for model reduction have appeared in the literature. Here, we present one based on Gramians and Balanced realizations that has some interesting theoretical interpretations.

Before we begin, let us take a look at the interpretation of the Gramians in terms of system properties. We start with the conceptually easier case of the observability Gramian where we formulate the following question: *Suppose that the linear system starts at $t = 0$ with initial condition x_0. Which states contribute more (or less) to the output energy in the interval $[0, \infty)$?*

Naturally, we assume that the system is stable so that the output energy is well defined. Then, with $y(t) = Ce^{At}x_0$, the output energy is

$$E_y = \|y\|_2^2 = \int_0^\infty x_0^\top e^{A^\top t} C^\top C e^{At} x_0 dt = x_0^\top N x_0$$

where N is the infinite interval $[0, \infty)$ observability Gramian. Bring in the SVD of N, so that $N = USU^\top$ (notice that N is symmetric, implying $U = V$). Then $E_y = (U^\top x_0)^\top S(U^\top x_0)$. Consider for a moment the (similarity) state transformation $\tilde{x} = U^\top x$. Then, in the transformed coordinates the contribution of the states to the output energy is ordered from the highest to the lowest. Furthermore, if the last $n - p$ singular values in S are less that ϵ, eliminating these states will change the output energy by no more than $\epsilon\|x_0\|^2$. Viewing this result in terms of operators between vector spaces, all that we have done is to map the unit ball in the domain to a set (ellipsoid) in the co-domain. Obviously, any states associated with domain directions that map to a small semi-axis in the co-domain can be ignored without introducing a significant error.

The dual interpretation of the controllability Gramian is a little more involved. A "natural" question to ask would be how much a unit-energy input contributes to the final state. For reasons that become apparent subsequently, it is convenient to consider inputs in $(-\infty, 0]$ and the final state as $x(0)$. In contrast to the observability case, the problem here is that our map $u \mapsto x(0)$ has a non-trivial null space. Instead, a more appropriate formulation would require the input to be constrained on the orthogonal complement of the null space of our operator. Equivalently, we may ask the question "what is the minimum energy input that is required to produce a unit-norm final state?" In our operator framework, we look for an ellipsoid in the domain that maps to the unit ball in the co-domain. Its interpretation is naturally the opposite from the previous case. Any co-domain directions that correspond to large semi-axis of the domain ellipsoid signify weekly controllable states; these can be ignored without introducing large errors. Thus, the problem at hand is to find the minimum norm input such that

$$x_0 = \int_{-\infty}^0 e^{-A\tau} Bu(\tau) d\tau$$

We have already computed the solution of this problem to be (with some minor adjustments for the interval)

$$u_{MN}(t) = B^\top e^{-A^\top t} M^{-1} x_0$$

where M is the infinite interval $[0, \infty)$ controllability Grammian. The energy of this optimal solution is

$$E_u = \int_{-\infty}^0 u^\top(\tau) u(\tau) d\tau = x_0^\top M^{-1} \left(\int_{-\infty}^0 e^{-A\tau} BB^\top e^{-A^\top \tau} d\tau \right) M^{-1} x_0 = x_0^\top M^{-1} x_0$$

Now, as in the observability case, large singular values of M correspond to small singular values of M^{-1}, meaning directions in the state space where unit-energy inputs have large effects. Consequently, through an SVD of M, we can rank the transformed states according to the energy they receive from the input.

[5] It is possible to obtain a generalization of this method to the multivariable case, but its numerical reliability is questionable (e.g., multiple poles).

The last step to take is to account for cases where weekly controllable modes are strongly observable and vice-versa. This would correspond to a combination of the previous two steps: The map from past inputs $(-\infty, 0]$ to states $x(0)$ and the map from states $x(0)$ to future outputs $[0, \infty)$. The composite map from the past inputs to future outputs is referred to as the system's Hankel operator and is an alternative way to characterize the system. What is interesting here is that we can rank the states in terms of their association with the energy transfer from past inputs to future outputs. For this, the following theorem is important:

4.6.1 Theorem: *For a stable minimal system $[A, B, C, D]$ there exists a similarity transformation T such that in the transformed coordinates the controllability and observability Gramians are equal and diagonal. Such a realization is referred to as* **Balanced Realization**. ▽▽

Proof: Let M, N be the two Gramians found as the solutions of the respective Lyapunov equations. Recall that a similarity transformation $x = T\bar{x}$ transforms the Gramians as

$$\bar{M} = T^{-1} M T^{-\top} \quad ; \quad \bar{N} = T^{\top} N T$$

We perform the transformation in two steps. In the first step we make the controllability Gramian equal to the identity. Since M is symmetric, its SVD has the form $M = U S_c U^{\top}$. Define the first transformation as $T_1 = U S_c^{1/2}$. In these first transformed coordinates, $\bar{M} = I$ and $\bar{N} = T_1^{\top} N T_1$.

In the second step, we perform an SVD on $\bar{N}$ to get $\bar{N} = V S_o V^{\top}$. Define the second transformation as $T_2 = V S_o^{-1/4}$. In the new coordinates, both Gramians are equal and diagonal:

$$\tilde{M} = \tilde{N} = S_o^{1/2}$$

The composite transformation is given by $T = T_1 T_2 = S_c^{1/2} U^{\top} V S_o^{-1/4}$. □□

In addition to an energy-related ranking of the states, balanced realizations have desirable numerical properties. That is, the system matrices are full and the magnitude of their entries is distributed more evenly than other canonical forms. The practical implication of this is that computations using balanced realizations are less prone to numerical problems than, say, controllable canonical forms (this is not a precise statement). It should also be mentioned that balanced realizations can be defined and computed for nonminimal stable systems. These, however, require a considerably more involved computation (a sequence of four transformations).

In a balanced realization, the model reduction problem is greatly simplified. In addition, the reduction error satisfies attractive system-gain bounds. The properties of this **balanced truncation** algorithm for model reduction are described by the following theorem.

4.6.2 Theorem: *(Balanced Truncation) Let $[A, B, C, D]$ be a balanced realization of a stable system. Denote the balanced Gramians by Σ and consider the partition*

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}$$

Also, partition the system matrices accordingly

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad ; \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad ; \quad C = (C_1, C_2)$$

and define the reduced order system as $[A_{11}, B_1, C_1, D]$. Then,

- *The reduced system is stable if Σ_1 and Σ_2 have no common diagonal entries.*
- *The "size" of the error system (γ_2-gain) is bounded by $2\,\mathrm{trace}(\Sigma_2)$.*

▽▽

The proof is rather involved and is omitted, but a few remarks on the application of the theorem are in order.

- The balanced truncation is applicable to the reduction of minimal stable systems of arbitrary dimensions, number of inputs and outputs. Although the minimality assumption can be relaxed, stability cannot. To handle unstable systems, a stable/anti-stable decomposition is performed first; the stable subsystem is then reduced while the anti-stable is preserved without change.

- During the reduction, care should be exercised to avoid splitting equal or similar singular values. Typically, the partitioning is selected at a point where the balanced Gramian singular values exhibit a "gap."

- The error bounds are fairly attractive but the reduction is not optimal in the γ_2-gain sense. Nevertheless, the algorithm produces reasonably good results.

- With some modifications, the balanced truncation can be used to perform a (very useful) frequency-weighted reduction. In practice this algorithm is quite efficient. Unfortunately, there is no general error bound for this approximation and the reduced order model is not guaranteed to be stable either.

- Special cases of weighted reduction are the relative and multiplicative reductions. In these cases, special formulae are availble and the results are similar to the un-weighted balanced truncation (stability guarantees and error bounds). They do, however, require the system to be square (same number of inputs and outputs).