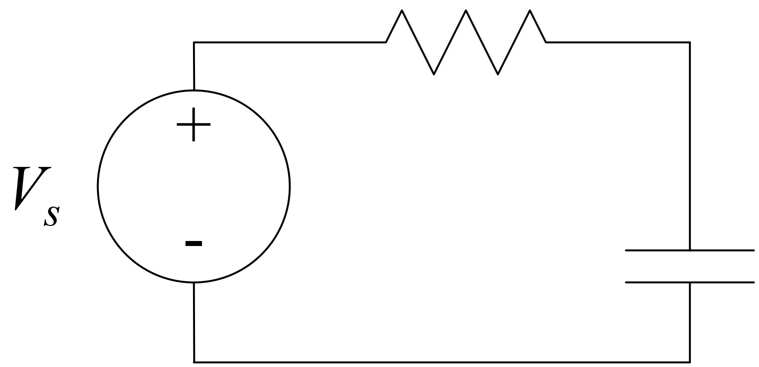
Solution of 1st-order linear ODEs

$$T\frac{dy}{dt}(t)+y(t)=x(t) \quad \Rightarrow \quad y(t)=e^{-\frac{t-t_0}{T}}y(t_0)+\frac{1}{T}\int_{t_0}^{t}e^{-\frac{t-\tau}{T}}x(\tau)d\tau$$

$$x(t)=x=const. \quad \Rightarrow \quad y(t)=e^{-\frac{t-t_0}{T}}y(t_0)+\left[1-e^{-\frac{t-t_0}{T}}\right]x=x-e^{-\frac{t-t_0}{T}}\left[x-y(t_0)\right]$$

Solutions of Simple RC Circuits



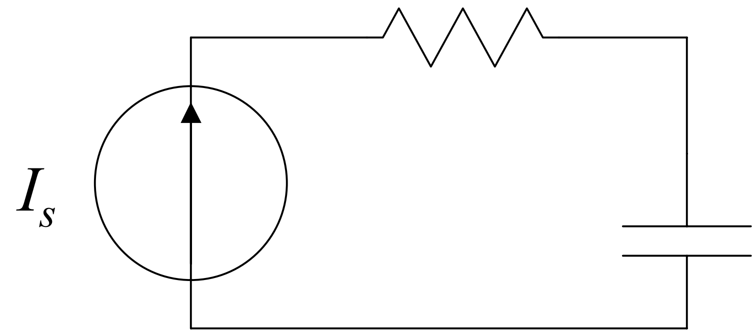
$$v_c+Ri=V_s; \quad i=C\frac{dv_c}{dt} \quad \Rightarrow$$

$$RC\frac{dv_c}{dt}+v_c=V_s \quad \Rightarrow$$

$$v_c(t)=e^{-\frac{t}{RC}}v_c(0)+\frac{1}{RC}\int_{0}^{t}e^{-\frac{t-\tau}{RC}}V_s(\tau)d\tau$$

$$const.V_s, \quad v_c(t)\rightarrow V_s$$

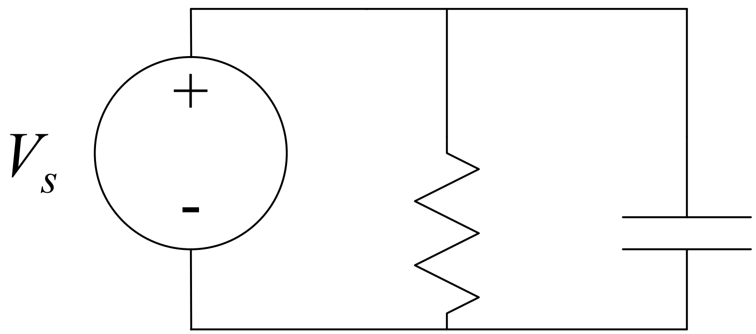
Solutions of Simple RC Circuits (cont.)



$$v_R = RI_s; \quad I_s = C\frac{dv_c}{dt} \quad \Rightarrow$$

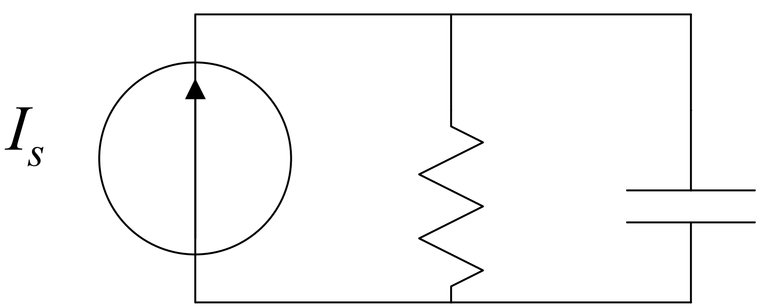
$$v_c(t) = v_c(0) + \frac{1}{C}\int_0^t I_s(\tau)d\tau$$

$$const.I_s, \quad v_c(t) = v_c(0) + \frac{I_s}{C}t \rightarrow \infty$$



$$v_R = v_c = V_s; \quad i_c = C\frac{dV_s}{dt} \quad \Rightarrow$$

$$discont.V_s, \quad i_c(t)\ not\ well-defined$$

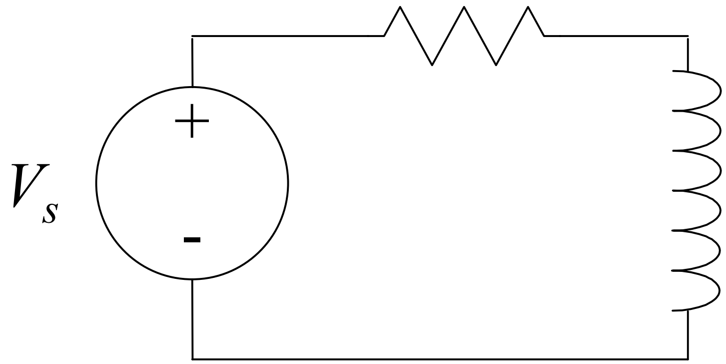


$$v_R = v_c \Rightarrow i_R = \frac{v_c}{R}; \quad i_R + i_c = I_s \Rightarrow RC\frac{dv_c}{dt} + v_c = RI_s$$

$$\Rightarrow \ v_c(t) = e^{-\frac{t}{RC}}v_c(0) + \frac{1}{RC}\int_0^t e^{-\frac{t-\tau}{RC}}RI_s(\tau)d\tau$$

$$const.\ I_s, \quad v_c(t) \rightarrow RI_s(= v_R)$$

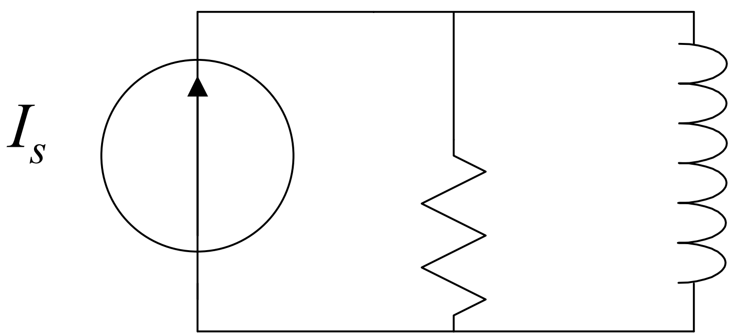
Solutions of Simple RL Circuits



$$V_s = RI_L + L\frac{di_L}{dt} \quad \Rightarrow$$

$$i_L(t) = e^{-\frac{R}{L}t} i_L(0) + \frac{R}{L}\int_0^t e^{-\frac{R}{L}(t-\tau)} \frac{1}{R} V_s(\tau)\,d\tau$$

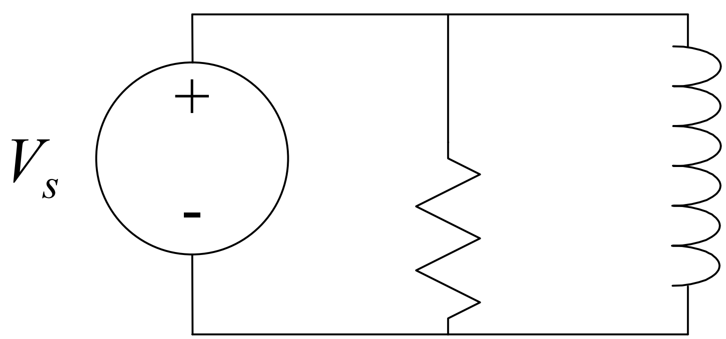
$$const.V_s, \quad i_L(t) \rightarrow \frac{V_s}{R}$$



$$V_R = V_L \Rightarrow Ri_R = L\frac{di_L}{dt}; \quad I_s = i_R + i_L = \frac{L}{R}\frac{di_L}{dt} + i_L$$

$$\Rightarrow \; i_L(t) = e^{-\frac{R}{L}t} i_L(0) + \frac{R}{L}\int_0^t e^{-\frac{R}{L}(t-\tau)} I_s(\tau)\,d\tau$$

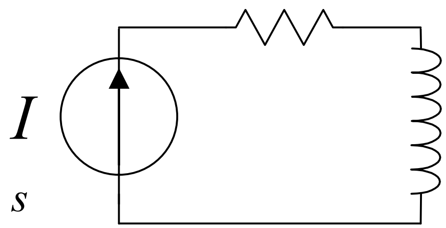
$$const.\, I_s, \quad i_L(t) \rightarrow I_s$$



$$v_L = V_s = L\frac{di_L}{dt} \quad \Rightarrow i_L(t) = i_L(0) + \frac{1}{L}\int_0^t V_s(\tau)\,d\tau$$

$$const.V_s, \quad i_c(t) = i_c(0) + \frac{V_s}{L} t \rightarrow \infty$$

Solutions of Simple RL Circuits (cont.)



$$i_L = I_s; \quad v_L = L\frac{dI_s}{dt} \quad \Rightarrow$$

$$discont.\ I_s, \quad v_L(t)\ not\ well-defined$$

Solutions of 2nd-Order linear ODEs

Notation

$$ODE: \quad \frac{d^2y}{dt^2}(t) + a\frac{dy}{dt}(t) + by(t) = x(t)$$

$$a > 0, \quad b > 0$$

$$char.eqn: \quad s^2 + as + b = 0$$

$$or \quad s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

$$or \quad (s - s_1)(s - s_2) = 0$$

initial conditions:

$$y(0) = y_0, \frac{dy}{dt}(0) = y_0'$$

Solution

$$y(t) = h_1(t)y_0 + h_2(t)y_0' + \int_0^t g(t-\tau)x(\tau)d\tau$$

Case 1 (*simple real roots*, $\zeta > 1$):

$$h_1(t) = \frac{s_2}{s_2 - s_1}e^{s_1 t} + \frac{s_1}{s_1 - s_2}e^{s_2 t}; \quad h_2(t) = \frac{-1}{s_2 - s_1}e^{s_1 t} + \frac{-1}{s_1 - s_2}e^{s_2 t};$$

$$g(t) = \frac{1}{s_1 - s_2}\left[e^{s_1 t} - e^{s_2 t}\right]$$

Case 2 (*double real root*, $\zeta = 1$):

$$h_1(t) = (1 - s_1 t)e^{s_1 t}; \quad h_2(t) = te^{s_1 t}; \quad g(t) = te^{s_1 t}$$

Case 3 (*complex roots*, $\zeta < 1$):

$$h_1(t) = \left(\cos\left[\omega_0\sqrt{1-\zeta^2}\ t\right] + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left[\omega_0\sqrt{1-\zeta^2}\ t\right]\right)e^{-\zeta\omega_0 t};$$

$$h_2(t) = g(t) = \frac{\sin\left[\omega_0\sqrt{1-\zeta^2}\ t\right]}{\omega_0\sqrt{1-\zeta^2}}e^{-\zeta\omega_0 t}$$