Problem 1:
Do Problems 7.30, 7.31 from the textbook.

7.30
The impulse response is the inverse Laplace of the transfer function:
\[ y_c(s) = \frac{1}{s+1} x(s) = \frac{1}{s+1} \]
⇒ \[ y_c(t) = e^{-t} u(t) \]
\[ y(n) = e^{-nT} u(nT) = \lambda^n u(n), \quad \lambda = e^{-T} \]
⇒ \[ y(z) = \frac{z}{z-\lambda} \]
For \( w(n) = \delta(n), \) i.e., \( w(z) = 1, \)
\[ w(z) = H(z) y(z) = H(z) \frac{z}{z-\lambda} \]
⇒ \[ H(z) = \frac{z-\lambda}{z} = 1 - \lambda z^{-1} \]
⇒ \[ H(e^{jwt}) = 1 - \lambda e^{-jwt}, \quad h(n) = \delta(n) - \lambda \delta(n-1) \]

7.31
Consider a test signal \( x_c(t) = \text{exp}(jwt), \) with \( w < \pi/T. \) Then, following the operations in Fig.P.7.31,
\[ x(n) = e^{jwtT} = (e^{jwT})^n \]
\[ y(n) = \frac{1}{2} y(n-1) + x(n) = H(z) \mid_{z=e^{jwT}} (e^{jwT})^n \] (since \( x \) is an exponential)
\[ = \frac{1}{1 - \frac{jwT}{2} z^{-1}} \mid_{z=e^{jwT}} (e^{jwT})^n = \frac{2}{2 - e^{-jwtT}} (e^{jwT})^n \]
\[ y_c(t) = \text{Lowpass} \left[ \frac{2}{2 - e^{-jwtT}} e^{jwtT} \right] = \frac{2}{2 - e^{-jwtT}} e^{jwt} \] (because \( w < \pi/T \))
⇒ \[ H(jw) = \frac{2}{2 - e^{-jwtT}}; \] (since, for an LTI system with an exponential input \( x_c(t) = e^{jwT}, y_c(t) = H(jw)e^{jwt} \))
Notice that from the last expression, the transfer function is \( H(s) = 2/(2-\exp(-sT)) \) which is not a finite dimensional system. Since the book does not specify the amplitude of the lowpass, in the reconstruction we assumed that the low-pass has the correct amplitude to recover the signal (i.e., \( T \)). If we assume an amplitude of 1, the transfer function must be divided by \( T \).

Problem 2:
Estimate the largest sampling interval \( T_s \) to allow perfect reconstruction of the signals \( (x*y) \) denotes convolution

1. \( \frac{\sin^2 2t}{t^2} \sin 3t \) Using the shortcuts, \( w_{max} = 2 + 2 + 3 = 7 \Rightarrow T_s < \frac{\pi}{7} \)
2. \[
\frac{\sin 2t}{t^2} \ast \sin 3t \quad w_{\text{max}} = \min(\infty, 3) = 3 \Rightarrow T_s < \frac{\pi}{3} \quad (1/t \text{ is not bandlimited})
\]
3. \[
\frac{\sin 3t \sin 2t}{2t} \quad w_{\text{max}} = 3 + 2 = 5 \Rightarrow T_s < \frac{\pi}{5}
\]
4. \[
\frac{\sin 3t}{t} \ast \sin 2t \quad w_{\text{max}} = \min(3, 2) = 2 \Rightarrow T_s = \frac{\pi}{2}
\]

Note: For (2), the Fourier of \(1/t\) is \(F\left\{\frac{1}{t}\right\} = \frac{\pi}{j} \text{sign}(w)\). This is a consequence of Duality, which can be briefly stated as \(FF = 2\pi R\), where, \(R\) denotes the reflection operation. Duality allows us to compute Fourier transforms for time functions that appear in the frequency column, e.g., \(1/jw\). (Verify!) Then, for example, \(F\left\{\frac{\sin(t)}{t}\right\} = \frac{1}{2\pi} \pi \text{sign}(w) \ast \text{pulse}(w, 1)\) which is not bandlimited. A similar computation appears in pp.311 of the textbook, but with a typo in Eqn. 4.42 (the integrant should be \(X(n)\)).

**Problem 3:**
Suppose that a continuous time signal \(x(t)\) is bandlimited to 1kHz and it is pre-processed by DT system with ideal sampling and reconstruction. The output of the discrete system is then processed by a CT system with transfer function \(H(s) = \frac{1}{s+1}\). Select a suitable sampling time \(T_s\) and find the discrete-time filter transfer function \(H_d(z)\) so that \(y(t) = x(t)\).

An exponential test input \(x(t) = e^{jwt}\), with \(w < 2000\pi\), will be reconstructed as \(v(t) = H_d(e^{jwT_s}) e^{jwt}\), provided that \(T_s < 0.5ms\). Then, after processing with the CT system, \(y(t) = H(jw)H_d(e^{jwT_s}) e^{jwt}\). Hence, the DT filter should be such that \(H(jw)H_d(e^{jwT_s}) = 1 \Rightarrow H_d(e^{jwT_s}) = jw + 1\). Or, \(H_d(z) = \frac{\log z}{T_s} + 1\). This cannot be implemented by a finite dimensional system, but can be approximated by one (e.g., via a Taylor expansion of log).