Problem 1. Use the function \( V = x^T Px \), \( P = \begin{bmatrix} 1 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} \), to find conditions on “\( a \)” such that the system \( \dot{x} = \begin{bmatrix} 0 & 1 \\ a & -2 \end{bmatrix} x \) is asymptotically stable.

1. Check \( P > 0: 1 > 0 \) and \( \det(P) > 0 \).
2. Check \( A^T P + PA = \begin{bmatrix} \frac{a}{2} + \frac{a}{2} & \frac{1}{2} + \frac{a}{2} \\ * & -\frac{3}{2} \end{bmatrix} < 0 \iff \begin{bmatrix} -\frac{a}{2} & -\frac{1}{2} - \frac{a}{2} \\ * & \frac{3}{2} \end{bmatrix} > 0 \iff \left\{ \begin{array}{l} a < 0 \\ a^2 + 5a + 1 < 0 \end{array} \right\} \iff -4.8 < a < -0.2 \)

Problem 2. Show that \( [A,B] \) is completely controllable if and only if \( [A-BK,B] \) is completely controllable, where \( K \) is any compatible matrix. \( [A,B] \) is c.c. iff for any set of \( n \) complex conjugate numbers \( G \) there exists \( K_0 \) such that \( \{ \text{eig}(A-BK_0) \} = G \). Define \( K_i = K_0 - K \); then \( \{ \text{eig}(A-BK-BK_i) \} = \{ \text{eig}(A-BK_0) \} = G \), which is equivalent to \( [A-BK,B] \) is c.c.

Problem 3. It is easy to show that if the eigenvalues of \( A \) have negative real parts, then the system \( [A,B,C,D] \) (standard state-space description) is BIBO stable. Give an example where the converse is not true, i.e., find a BIBO stable system whose realization has system matrix \( A \) with some eigenvalues with positive real parts.

The poles of \( G(s) = C(sI - A)^{-1}B + D \) are a subset of the eigenvalues of \( A \); any uncontrollable or unobservable eigenvalues of \( A \) will not be poles of \( G(s) \). Let, \( A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \), \( B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), \( C = [1 \ 0], D = 0 \); Then, \( G(s) = \frac{1}{s+1} \). \( G(s) \) is BIBO stable but \( \text{eig}(A) = \{-1, 1\} \).

Problem 4. Suppose \( [A,B,C,D], [F,G,H,J] \) are two realizations of the same \( n \)-th order transfer function \( P(s) \). Determine if the following statements are True or False. If False, provide a corrected or completed version:

- \( [A,B,C,D], [F,G,H,J] \) are algebraically equivalent (related by a coordinate transformation), if \( \text{dim}(A) = \text{dim}(F) = n \).
- \( [A,B,C,D], [F,G,H,J] \) are exponentially stable if \( P(s) \) is BIBO stable and \( \text{dim}(A) = \text{dim}(F) = n \).
- \( [A,B,C,D], [F,G,H,J] \) are algebraically equivalent if they are both controllable and \( \text{dim}(A) = \text{dim}(F) = n \).
- \( [A,B,C,D], [F,G,H,J] \) are NOT algebraically equivalent (related by a coordinate transformation), if \( \text{dim}(A) > \text{dim}(F) \).

1. False; A true statement would have the additional condition \( [A,B,C,D] \) is minimal.
2. False; A true statement would have the additional condition \( [A,B,C,D] \) is minimal.
3. False; A true statement would have the additional condition \( [A,B,C,D] \) is observable.
4. True. (Algebraic equivalence preserves the system order.)